

Određeni integrali

Osobine određenih integrala su:

$$1_0 \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$2_0 \int_a^a f(x) dx = 0$$

$$3_0 \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$4_0 \int_a^b [f_1(x) + f_2(x) - f_3(x)] dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx - \int_a^b f_3(x) dx$$

$$5_0 \int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx, \text{ gdje je } \alpha \text{ konstanta}$$

Određene integrale često računati pomoću Njuth-Lejbcove formule

$$\int_a^b f(x) dx = \int_a^b f(x) dx \Big|_a^b = F(x) \Big|_a^b = F(b) - F(a)$$

gdje je $F'(x) = f(x)$

Izračunajte integrale

a) $\int_2^3 3x^2 dx$; b) $\int_0^4 (1+e^{\frac{x}{4}}) dx$; c) $\int_{-1}^7 \frac{dt}{\sqrt{3t+4}}$;

d) $\int_0^{\frac{\pi}{2a}} (x+3) \sin ax$

Rj. a) $\int_2^3 3x^2 dx = 3 \int_2^3 x^2 dx = 3 \cdot \frac{x^3}{3} \Big|_2^3 = x^3 \Big|_2^3 = 3^3 - 2^3 = 27 - 8 = 19$

b) $\int_0^4 (1+e^{\frac{x}{4}}) dx = \int_0^4 dx + \int_0^4 e^{\frac{x}{4}} dx = \int_0^4 dx + 4 \int_0^4 e^{\frac{x}{4}} d(\frac{x}{4}) =$
 $= x \Big|_0^4 + 4 e^{\frac{x}{4}} \Big|_0^4 = (4-0) + 4(e^{\frac{4}{4}} - e^{\frac{0}{4}}) = 4 + 4e - 4 = 4e$

c) $\int_{-1}^7 \frac{dt}{\sqrt{3t+4}} = \int_{-1}^7 (3t+4)^{-\frac{1}{2}} dt = \left| \begin{array}{l} d(3t+4) = 3 dt \\ dt = \frac{1}{3} d(3t+4) \end{array} \right| =$
 $= \frac{1}{3} \int_{-1}^7 (3t+4)^{-\frac{1}{2}} d(3t+4) = \frac{2}{3} (3t+4)^{\frac{1}{2}} \Big|_{-1}^7 = \frac{2}{3} (\sqrt{25} - \sqrt{1}) = \frac{8}{3}$

d) $\int_0^{\frac{\pi}{2a}} (x+3) \sin ax dx = \left| \begin{array}{l} u = x+3 \quad dv = \sin ax dx \\ du = dx \quad v = \frac{1}{a} \int \sin ax d(ax) = -\frac{1}{a} \cos ax \end{array} \right| =$
 $= -\frac{1}{a} (x+3) \cos ax \Big|_0^{\frac{\pi}{2a}} + \frac{1}{a} \int_0^{\frac{\pi}{2a}} \cos ax dx = -\frac{1}{a} \left[\underbrace{\left(\frac{\pi}{2a} + 3\right)}_{=0} \underbrace{\cos \frac{\pi}{2}}_{=-1} - 3 \underbrace{\cos 0}_{=-1} \right] +$
 $+ \frac{1}{a} \cdot \frac{1}{a} \int_0^{\frac{\pi}{2a}} \cos ax d(ax) = \frac{3}{a} + \frac{1}{a^2} \underbrace{\sin ax \Big|_0^{\frac{\pi}{2a}}}_{\sin \frac{\pi}{2} - \sin 0} = \frac{3}{a} + \frac{1}{a^2} = \frac{1+3a}{a^2}$

Zadaci za vježbu

$$\textcircled{1}_0 \int_1^5 \frac{dx}{3x-2}$$

$$\textcircled{2}_0 \int_0^1 \frac{dz}{(2z+1)^3}$$

$$\textcircled{3}_0 \int_1^2 \frac{dt}{t^2+5t+4}$$

$$\textcircled{4}_0 \int_0^2 \frac{x+3}{x^2+4} dx$$

$$\textcircled{5}_0 \int_{-a}^a x \cos \frac{x}{a} dx$$

$$\textcircled{6}_0 \int_0^\pi \cos \frac{x}{2} \cos \frac{3x}{2} dx$$

$$\textcircled{7}_0^* \int_{-\pi}^\pi x \sin x \cos x dx$$

$$\textcircled{8}_0 \int_1^e (1+\ln y)^2 dy$$

Rešenja:

$$1_0 \frac{\ln 13}{3}$$

$$2_0 \frac{2}{9}$$

$$3_0 \frac{1}{3} \ln \frac{5}{4}$$

$$4_0 \frac{3\pi}{8} + \frac{\ln 2}{2}$$

$$5_0 0$$

$$6_0 0$$

$$7_0 -\frac{\pi}{2}$$

$$8_0 2e-1$$

Zamena promjenjivih u određenom integralu

$$\int_a^b f(x) dx = \left| \begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t) dt \end{array} \right. \quad \begin{array}{l} x=a \Rightarrow a = \varphi(\alpha) \Rightarrow t = \alpha \\ x=b \Rightarrow b = \varphi(\beta) \Rightarrow t = \beta \end{array} \left. \right|$$
$$= \int_{\alpha}^{\beta} f(\varphi(t)) \varphi'(t) dt = \int_{\alpha}^{\beta} F(t) dt$$

Ⓝ Iračunati integrale

a) $\int_0^5 \frac{x dx}{\sqrt{1+3x}}$; b) $\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}}$; c) $\int_1^{\sqrt{3}} \frac{(x^3+1) dx}{x^2 \sqrt{4-x^2}}$; d) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Rj. a) $\int_0^5 \frac{x dx}{\sqrt{1+3x}} = \left| \begin{array}{l} 1+3x = t^2 \\ \sqrt{1+3x} = t \\ 3x = t^2 - 1 \\ x = \frac{t^2 - 1}{3} \end{array} \right. \quad \left. \begin{array}{l} 3 dx = 2t dt \\ dx = \frac{2}{3} t dt \\ x|_0^5 \Rightarrow t|_1^4 \end{array} \right| = \int_1^4 \frac{\frac{t^2-1}{3} \cdot \frac{2}{3} t dt}{t} =$

$$= \frac{2}{9} \int_1^4 (t^2 - 1) dt = \frac{2}{9} \left(\frac{t^3}{3} \Big|_1^4 - t \Big|_1^4 \right) = \frac{2}{9} \left(\frac{1}{3} (64 - 1) - (4 - 1) \right) =$$
$$= \frac{2}{9} \left(\frac{63}{3} - 3 \right) = \frac{2}{9} (21 - 3) = 4$$

b) $\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}} = \left| \begin{array}{l} e^x = t \\ x = \ln t \\ dx = \frac{dt}{t} \end{array} \right. \quad \left. \begin{array}{l} e^{-x} = t^{-1} = \frac{1}{t} \\ x|_{\ln 2}^{\ln 3} \Rightarrow t|_2^3 \end{array} \right| = \int_2^3 \frac{\frac{dt}{t}}{t - \frac{1}{t}} = \int_2^3 \frac{\frac{dt}{t}}{\frac{t^2 - 1}{t}} =$

$$= \int_2^3 \frac{dt}{t^2 - 1} = \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| \Big|_2^3 = \frac{1}{2} \left(\ln \frac{2}{4} - \ln \frac{1}{3} \right) = \frac{1}{2} \cdot \ln \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\ln \frac{3}{2}}{2}$$

$$\begin{aligned}
 \text{c) } \int_1^{\sqrt{3}} \frac{(x^3+1) dx}{x^2 \sqrt{4-x^2}} &= \left. \begin{array}{l} x = 2 \sin t \\ dx = 2 \cos t dt \\ x^3 = 8 \sin^3 t \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} \end{array} \right|_1^{\sqrt{3}} \Rightarrow \left. t \right|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(8\sin^3 t + 1) 2 \cos t dt}{4 \sin^2 t \sqrt{4 \cos^2 t}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^3 t + 1}{4 \sin^2 t} dt = \\
 &= 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin t dt + \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -2 \cos t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{4} \cot t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \\
 &= -2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2} \right) - \frac{1}{4} \left(\frac{\sqrt{3}}{3} - \sqrt{3} \right) = \frac{7}{2\sqrt{3}} - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{d) } \int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x} &= \left. \begin{array}{l} \text{tg } \frac{x}{2} = z \\ \cos x = \frac{1-z^2}{1+z^2} \\ dx = \frac{2dz}{1+z^2} \end{array} \right|_0^{\frac{\pi}{2}} \Rightarrow \left. z \right|_0^1 = \\
 &= \int_0^1 \frac{\frac{2dz}{1+z^2}}{2 + \frac{1-z^2}{1+z^2}} = 2 \int_0^1 \frac{dz}{z^2 + 3} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{z}{\sqrt{3}} \Big|_0^1 = \\
 &= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}
 \end{aligned}$$

(#) Dokazati da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

dok za neparnu f-ju $f(x)$ vrijedi $\int_{-a}^a f(x) dx = 0$.

Rj. Prvo rastavimo interval $[-a, a]$ na dva dijela $[-a, 0]$ i $[0, a]$

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad \dots (*)$$

Pozmatrajmo sad $\int_{-a}^0 f(x) dx$. Ako uvedemo smjenu

$x = -z$ imamo da je $dx = -dz$ i $z_1 = a$ za $x_1 = -a$,

$z_2 = 0$ za $x_2 = 0$

$$\int_{-a}^0 f(x) dx = - \int_a^0 f(-z) dz = \int_0^a f(-z) dz = \int_0^a f(-x) dx$$

novi promij
x=z

Prema tome (*) sad postaje

$$I = \int_{-a}^a f(x) dx = \int_0^a f(-x) dx + \int_0^a f(x) dx = \int_0^a [f(x) + f(-x)] dx$$

Za parnu f-ju znamo da $f(-x) = f(x)$ dok je za neparnu f-ju $f(-x) = -f(x)$. Prema tome

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{ako je } f(x) \text{ parna f-ja} \\ 0, & \text{ako je } f(x) \text{ neparna f-ja} \end{cases}$$

Znamo da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx,$$

dok za neparnu f-ju $f(x)$ vrijedi: $\int_{-a}^a f(x) dx = 0$.
Iskoristiti ovu osobinu i izračunati sljedeće integrale:

a) $\int_{-\sqrt{5}}^{\sqrt{5}} (3x - 2x^5) dx$

b) $\int_{-\pi}^{\pi} \sin^7 2x dx$

c) $\int_3^{-3} t^8 \arcsin t dt$

d) $\int_{-2}^2 \frac{x^5 + 7x^4 + x^3 - 5x^2 - 2}{x^3 + x} dx$

Rj.

a) $f(x) = 3x - 2x^5$

$$f(-x) = 3(-x) - 2(-x)^5 = -3x + 2x^5 = -(3x - 2x^5) = -f(x)$$

$$\int_{-\sqrt{5}}^{\sqrt{5}} (3x - 2x^5) dx = \left| \begin{array}{l} \text{primjetimo} \\ \text{da je} \\ 3x - 2x^5 \text{ neparna} \\ \text{f-ja} \end{array} \right| = 0$$

b) $f(x) = \sin^7 2x \Rightarrow f(-x) = (\sin 2(-x))^7 = (-\sin 2x)^7 = -\sin^7 2x = -f(x)$

Kako je $\sin^7 2x$ neparna f-ja $\int_{-\pi}^{\pi} \sin^7 2x dx = 0$

c) $\int_3^{-3} t^8 \arcsin t dt = 0$ ZAŠTO? OBJASNITI!

d) $\int_{-2}^2 \frac{x^5 + 7x^4 + x^3 - 5x^2 - 2}{x^3 + x} dx = \int_{-2}^2 \frac{x^2(x^3+x)}{x^3+x} dx + \int_{-2}^2 \frac{7x^4 - 5x^2 - 2}{x^3+x} dx =$
 $= \int_{-2}^2 x^2 dx + 0 = 2 \int_0^2 x^2 dx = 2 \left. \frac{x^3}{3} \right|_0^2 = \frac{16}{3}$

Zadaci za vježbu

Izračunati integrale

1₀ $\int_0^1 \frac{x^2 dx}{(x+1)^4}$ uvođenjem smjene $x+1=z$.

2₀ $\int_0^{\ln 2} \sqrt{e^x - 1} dx$ uvođenjem smjene $\sqrt{e^x - 1} = t$.

3₀ $\int_{\sqrt{3}}^{\sqrt{7}} \frac{x^3 dx}{\sqrt[3]{(x^2+1)^2}}$ uvođenjem smjene $z = x^2 + 1$.

4₀ $\int_1^e \frac{\sqrt[4]{1+\ln x}}{x} dx$ uvođenjem smjene $t = 1 + \ln x$.

5₀ $\int_{-3}^3 x^2 \sqrt{9-x^2} dx$ uvođenjem smjene $x = 3 \cos \varphi$

6₀ $\int_5^1 \frac{t dt}{\sqrt{5+4t}}$ 7₀ $\int_0^{\frac{\pi}{4}} \frac{1+t \varphi^2}{1+t \varphi} d\varphi$ 8₀ $\int_{\ln 3}^0 \frac{1-e^x}{1+e^x} dx$

9₀ $\int_{-1}^0 \frac{dx}{1+\sqrt[3]{x+1}}$ 10₀ $\int_0^8 \sqrt{\frac{x}{6-x}} dx$ 11₀ $\int_0^{\frac{\pi}{2}} \sin^3 \varphi \sqrt{\cos \varphi} d\varphi$

Rješenja:

1₀ $\frac{1}{24}$ 2₀ $\frac{4-\pi}{2}$ 3₀ 3 4₀ $0,8(2\sqrt[4]{2}-1)$ 5₀ $\frac{81\pi}{8}$ 6₀ $-\frac{17}{6}$

7₀ $\ln 2$ 8₀ $\ln \frac{4}{3}$ 9₀ $\frac{3}{2}(\ln 4 - 1)$ 10₀ $\frac{3(\pi-2)}{2}$

11₀ $8/21$ (uvodimo smjenu $x = 6 \sin^2 t$)

Izabrani Zadaci za vježbu sa rješenjima

(iz lekcije Računje određenih integrali i

Smjena promjenjivih u određenim integralima)

$$\textcircled{1} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\cos^2 x} = \operatorname{tg} x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \operatorname{tg} \frac{\pi}{3} - \operatorname{tg} \frac{\pi}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

$$\textcircled{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x \, dx = -\cos x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -(\cos \frac{\pi}{3} - \cos \frac{\pi}{4}) = -(\frac{1}{2} - \frac{\sqrt{2}}{2}) = -\frac{1-\sqrt{2}}{2}$$

$$\textcircled{3} \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \arcsin \frac{\sqrt{3}}{2} - \arcsin \frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\textcircled{4} \int_a^b x^m \, dx = \frac{x^{m+1}}{m+1} \Big|_a^b = \frac{1}{m+1} (b^{m+1} - a^{m+1})$$

$$\textcircled{5} \int_0^1 (e^x - 1)^4 e^x \, dx = \left| \begin{array}{l} e^x - 1 = t \\ e^x \, dx = dt \\ x=0 \Rightarrow t=0 \\ x=1 \Rightarrow t=e-1 \end{array} \right| = \int_0^{e-1} t^4 \, dt = \frac{t^5}{5} \Big|_0^{e-1} = \frac{1}{5} (e-1)^5$$

$$\textcircled{6} \int_2^9 \sqrt[3]{x-1} \, dx = \left| \begin{array}{l} x-1 = t^3 \\ dx = 3t^2 \, dt \\ x=2 \Rightarrow t=1 \\ x=9 \Rightarrow t=2 \end{array} \right| = \int_1^2 \sqrt[3]{t^3} \cdot 3t^2 \, dt = 3 \int_1^2 t^3 \, dt = \frac{3}{4} t^4 \Big|_1^2 = \frac{3}{4} (16-1) = \frac{45}{4}$$

$$\textcircled{7} \int_0^2 \frac{\sqrt{e^x - 1}}{1 + 3e^{-x}} \, dx = \int_0^2 \frac{e^x \sqrt{e^x - 1}}{e^x + 3} \, dx = \left| \begin{array}{l} e^x - 1 = t^2 \\ e^x \, dx = 2t \, dt \\ x=0 \Rightarrow t=0 \\ x=\ln 5 \Rightarrow t=2 \\ e^x = t^2 + 1 \end{array} \right| = \int_0^2 \frac{\sqrt{t^2} \cdot 2t}{t^2 + 1 + 3} \, dt$$
$$= 2 \int_0^2 \frac{t^2 + 4 - 4}{t^2 + 4} \, dt = 2 \int_0^2 dt - 2 \int_0^2 \frac{4}{t^2 + 4} \, dt = 2t \Big|_0^2 - 8 \cdot \frac{1}{2} \operatorname{arctg} \frac{t}{2} \Big|_0^2 =$$
$$= 4 - 4 (\operatorname{arctg} 1 - \operatorname{arctg} 0) = 4 - 4 \cdot \frac{\pi}{4} = 4 - \pi$$

Osobine odredenih integrala

a) $\int_a^a f(x) dx = 0$

b) $\int_a^b f(x) dx = - \int_b^a f(x) dx$

c) $\int_a^b c \cdot f(x) dx = c \int_a^b f(x) dx$

d) $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

e) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad \forall x$

8) $\int_0^{\sqrt{7}} \frac{dx}{7+x^2} = \frac{1}{\sqrt{7}} \arctan \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}} (\arctan \frac{\sqrt{7}}{\sqrt{7}} - \arctan \frac{0}{\sqrt{7}}) = \frac{1}{\sqrt{7}} \cdot \frac{\pi}{4} = \frac{\sqrt{7}\pi}{28}$

9) $\int_0^{1/2} \sqrt{1-x^2} dx = \begin{cases} x = \sin t \\ dx = \cos t dt \\ x=0 \Rightarrow \sin t = 0 \Rightarrow t=0 \\ x=1/2 \Rightarrow \sin t = 1/2 \Rightarrow t=\pi/6 \end{cases} = \int_0^{\pi/6} \sqrt{1-\sin^2 t} \cos t dt = \int_0^{\pi/6} \cos^2 t dt$
 $= \int_0^{\pi/6} \cos^2 t dt = \frac{1}{2} \int_0^{\pi/6} (1 + \cos 2t) dt = \frac{1}{2} t \Big|_0^{\pi/6} + \frac{1}{4} \sin 2t \Big|_0^{\pi/6}$
 $= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{2\pi + 3\sqrt{3}}{24}$
kada je $\sin^2 t + \cos^2 t = 1$
 $1 = \cos^2 t + \sin^2 t$
 $\cos 2t = \cos^2 t - \sin^2 t$

10) $\int_0^4 \frac{dx}{1+\sqrt{2x+1}}$ uputa: smjena $2x+1=t^2$
 j: $2 - \ln 2$

11) $\int_0^1 \frac{dx}{\sqrt{2-x^2+x}}$ uputa: $-x^2+x+2 = \dots$
 $\dots = \frac{9}{4} - (x-\frac{1}{2})^2$
 $x-1 = \frac{3}{2}t$
 j: $2 \arcsin \frac{1}{3}$

12) $\int_1^e x \ln x dx = \begin{cases} u = \ln x & dv = x dx \\ du = \frac{dx}{x} & v = \frac{x^2}{2} \end{cases} = \frac{1}{2} x^2 \ln x \Big|_1^e - \frac{1}{2} \int_1^e x^2 \cdot \frac{dx}{x} =$
 $= \frac{1}{2} (e^2 \ln e - 1^2 \ln 1) - \frac{1}{2} \int_1^e x dx = \frac{1}{2} e^2 - \frac{1}{2} \cdot \frac{1}{2} x^2 \Big|_1^e =$
 $= \frac{1}{2} e^2 - \frac{1}{4} (e^2 - 1) = \frac{1}{4} e^2 + \frac{1}{4}$

Izračunati integral $\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1 = (\sqrt{x}-2)^2+1$

$$\int_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1} dx = \left| \begin{array}{l} x=t^2 \\ dx=2t dt \\ x=1 \Rightarrow t=1 \\ x=4 \Rightarrow t=2 \end{array} \right| = \int_1^2 \frac{t+2}{(t-2)^2+1} \cdot 2t dt =$$

$$= 2 \int_1^2 \frac{t^2+2t}{t^2-4t+5} dt = 2 \int_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5} dt = 2 \int_1^2 \frac{t^2-4t+5}{t^2-4t+5} dt +$$

$$+ 2 \int_1^2 \frac{6t-5}{t^2-4t+5} dt = 2 \int_1^2 dt + 2 \cdot 3 \int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5} dt$$

$$\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5} dt = \int_1^2 \frac{2t-4+4-\frac{5}{3}}{t^2-4t+5} dt = \int_1^2 \frac{2t-4}{t^2-4t+5} dt + \frac{7}{3} \int_1^2 \frac{dt}{t^2-4t+5}$$

$$\int_1^2 dt = t \Big|_1^2 = 2-1=1, \quad \int_1^2 \frac{2t-4}{t^2-4t+5} dt = \left| \begin{array}{l} t^2-4t+5 = s \\ (2t-4)dt = ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1 \end{array} \right| = \int_2^1 \frac{ds}{s} = \ln|s| \Big|_2^1$$

$$\int_1^2 \frac{dt}{t^2-4t+5} = \int_1^2 \frac{dt}{(t-2)^2+1} = \left| \begin{array}{l} t-2 = s \\ dt = ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0 \end{array} \right| = \int_{-1}^0 \frac{ds}{s^2+1} = \arctg s \Big|_{-1}^0 = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2 \cdot 1 + 6 \left(-\ln 2 + \frac{7}{3} \cdot \frac{\pi}{4} \right) = 2 - 6 \ln 2 + \frac{7\pi}{2} \approx 8,8367$$

traženo je

Ⓝ Izračunati integral $\int_0^{\frac{\pi}{4}} \sin^5 x \cos^7 x dx$

$$Rj: \int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^7 x dx = \int_0^{\frac{\pi}{4}} \sin^4 x \cdot \cos^6 x \cdot \cos x dx = \left. \begin{array}{l} \sin x = t \\ \cos x dx = dt \\ x=0 \Rightarrow t=0 \\ x=\frac{\pi}{4} \Rightarrow t=\frac{\sqrt{2}}{2} \\ \cos^6 x = (\cos^2 x)^3 = \\ = (1 - \sin^2 x)^3 = (1 - t^2)^3 \end{array} \right| =$$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^4 (1-t^2)^3 dt =$$

$$\begin{array}{c} 1 \\ 1 \ 1 \\ 1 \ 2 \ 1 \\ 1 \ 3 \ 3 \ 1 \end{array}$$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^4 (1 - 3t^2 + 3t^4 - t^6) dt = \int_0^{\frac{\sqrt{2}}{2}} (t^4 - 3t^6 + 3t^8 - t^{10}) dt =$$

$$= \left. \frac{1}{5} t^5 - \frac{3}{7} t^7 + \frac{3}{9} t^9 - \frac{1}{11} t^{11} \right|_0^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{1}{5} \cdot \frac{1}{16} - \frac{3}{7} \cdot \frac{1}{128} + \frac{3}{9} \cdot \frac{1}{64} - \frac{1}{11} \cdot \frac{1}{256} =$$

$$= \frac{1}{3 \cdot 2^4} - \frac{3}{128} + \frac{3}{5 \cdot 64} - \frac{1}{3 \cdot 256} = \frac{5 \cdot 2^4 - 3 \cdot 3 \cdot 5 \cdot 2 + 3 \cdot 3 \cdot 2^2 - 5}{3 \cdot 5 \cdot 2^8}$$

$$= \frac{80 - 90 + 36 - 5}{3 \cdot 5 \cdot 2^8} = \frac{21}{1 \cdot 5 \cdot 2^8} = \frac{7}{1280} \quad \text{traženo rješenje}$$

Izračunati integral $\int_{-1}^1 \frac{2x^3 - 7x + 4}{\sqrt{x^2 + 3}} dx$

Rj. Metoda Ostrogradskog:

$$\int \frac{2x^3 - 7x + 4}{\sqrt{x^2 + 3}} dx = (ax^2 + bx + c) \sqrt{x^2 + 3} + \lambda \int \frac{dx}{\sqrt{x^2 + 3}} \quad \Big| \frac{d}{dx}$$

$$\frac{2x^3 - 7x + 4}{\sqrt{x^2 + 3}} = (2ax + b) \sqrt{x^2 + 3} + (ax^2 + bx + c) \frac{dx}{\sqrt{x^2 + 3}} + \frac{\lambda}{\sqrt{x^2 + 3}}$$

$$2x^3 - 7x + 4 = (2ax + b)(x^2 + 3) + (ax^2 + bx + c)x + \lambda$$

$$\underline{2x^3 - 7x + 4} = \underline{2ax^3} + \underline{bx^2} + \underline{6ax + 3b} + \underline{ax^3 + bx^2 + cx} + \lambda$$

x^3 : $2a + a = 2 \Rightarrow 3a = 2$
 $a = \frac{2}{3}$

x^1 : $6a + c = -7 \Rightarrow 6 \cdot \frac{2}{3} + c = -7$

x^2 : $b + b = 0 \Rightarrow b = 0$

x^0 : $3b + \lambda = 4$
 $\lambda = 4$

$4 + c = -7$
 $c = -11$

Prava točka:

$$\int \frac{2x^3 - 7x + 4}{\sqrt{x^2 + 3}} dx = \left(\frac{2}{3}x^2 - 11\right) \sqrt{x^2 + 3} + 4 \int \frac{dx}{\sqrt{x^2 + 3}} =$$

$$= \frac{2}{3}x^2 \sqrt{x^2 + 3} - 11\sqrt{x^2 + 3} + 4 \ln|x + \sqrt{x^2 + 3}| + C$$

Prava točka

$$\int_{-1}^1 \frac{2x^3 - 7x + 4}{\sqrt{x^2 + 3}} dx = \frac{2}{3}x^2 \sqrt{x^2 + 3} \Big|_{-1}^1 - 11\sqrt{x^2 + 3} \Big|_{-1}^1 + 4 \ln|x + \sqrt{x^2 + 3}| \Big|_{-1}^1 =$$

$$= \frac{2}{3}(2 - 2) - 11(2 - 2) + 4(\ln|1 + 2| - \ln|-1 + 2|) =$$

$$= 4(\ln 3 - \ln 1) = 4 \ln 3$$

traženi
rezultat

Ⓝ) Izračunati:

$$\int_3^4 \frac{6x+8}{x^2+x-6} dx$$

Rj.

$$\frac{6x+8}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} \quad / (x-2)(x+3)$$

$$D = 1 + 24 = 25$$

$$x_{1,2} = \frac{-1 \pm 5}{2}$$

$$x_1 = -3 \quad x_2 = 2$$

$$x^2+x-6 = (x-2)(x+3)$$

$$6x+8 = A(x+3) + B(x-2)$$

$$6x+8 = (A+B)x + (3A-2B)$$

$$A+B = 6 \quad / \cdot 2$$

$$3A-2B = 8$$

$$2A+2B = 12$$

$$+ 3A-2B = 8$$

$$5A = 20$$

$$A = 4$$

$$A+B = 6$$

$$4+B = 6$$

$$B = 2$$

$$\int \frac{6x+8}{x^2+x-6} dx = \int \left(\frac{4}{x-2} + \frac{2}{x+3} \right) dx = 4 \int \frac{dx}{x-2} + 2 \int \frac{dx}{x+3} =$$

$$= 4 \ln|x-2| + 2 \ln|x+3| + C$$

$$\int_3^4 \frac{6x+8}{x^2+x-6} dx = 4 \ln|x-2| \Big|_3^4 + 2 \ln|x+3| \Big|_3^4 = 4(\ln 2 - \ln 1) +$$

$$+ 2(\ln 7 - \ln 6) = 4 \ln 2 + 2 \ln \frac{7}{6} = \ln 2^4 + \ln \left(\frac{7}{6} \right)^2$$

$$= \ln \frac{7^2}{2^2 \cdot 3^2} \cdot 2^4 = \ln \frac{49 \cdot 4}{9} = \ln \frac{196}{9}$$

$$\int_3^4 \frac{6x+8}{x^2+x-6} dx = \ln \frac{196}{9}$$

traženo rešenje

(#) Izračunati integral $I = \int_0^1 \arcsin \frac{x}{2} dx$.

Rj.

$$\int \arcsin \frac{x}{2} dx = \left| \begin{array}{l} \frac{x}{2} = t \\ \frac{1}{2} dx = dt \\ dx = 2 dt \end{array} \right| = 2 \int \arcsin t dt = \left| \begin{array}{ll} u = \arcsin t & dv = dt \\ du = \frac{dt}{\sqrt{1-t^2}} & v = t \end{array} \right| =$$

$$= 2 \left(t \arcsin t - \int \frac{t}{\sqrt{1-t^2}} dt \right) = 2t \arcsin t - \int \frac{2t dt}{\sqrt{1-t^2}} \quad (**)$$

$$\int \frac{-2t dt}{\sqrt{1-t^2}} = \left| \begin{array}{l} 1-t^2 = s \\ -2t dt = ds \end{array} \right| = \int \frac{ds}{\sqrt{s}} = \int s^{-\frac{1}{2}} ds = \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{s} + C =$$

$$= 2\sqrt{1-t^2} + C$$

$$(**) \quad 2t \arcsin t + 2\sqrt{1-t^2} + C = x \arcsin \frac{x}{2} + 2\sqrt{1-\frac{x^2}{4}} + C$$

$$\int_0^1 \arcsin \frac{x}{2} dx = x \arcsin \frac{x}{2} \Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}} \Big|_0^1 = \arcsin \frac{1}{2} + \left(2\sqrt{1-\frac{1}{4}} - 2 \right) =$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2} - 2 = \frac{\pi}{6} + \sqrt{3} - 2$$

Nepravi integrali

Nepravi integral u granicama od a do $+\infty$ je oblika

$$I = \int_a^{+\infty} f(x) dx$$

Rješavamo ga na sledeći način:

$$\int_a^{+\infty} f(x) dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) dx$$

Ako postoji

$$\lim_{b \rightarrow +\infty} \int_a^b f(x) dx$$

kažemo da integral konvergira ili da postoji nepravi integral, a ako limes ne postoji (kao realan broj), kažemo da integral divergira ili da nepravi integral ne postoji.

1) Izračunati:

$$\begin{aligned} \text{a) } \int_1^{+\infty} \frac{dx}{x^4} &= \lim_{a \rightarrow +\infty} \int_1^a \frac{dx}{x^4} = \lim_{a \rightarrow +\infty} \int_1^a x^{-4} dx = \lim_{a \rightarrow +\infty} \left. \frac{x^{-3}}{-3} \right|_1^a = \\ &= -\frac{1}{3} \lim_{a \rightarrow +\infty} \left. \frac{1}{x^3} \right|_1^a = -\frac{1}{3} \lim_{a \rightarrow +\infty} \left(\frac{1}{a^3} - 1 \right) = \left(-\frac{1}{3} \right) (-1) = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{b) } \int_1^{+\infty} \frac{dx}{\sqrt{x}} &= \lim_{b \rightarrow +\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow +\infty} \int_1^b x^{-\frac{1}{2}} dx = \lim_{b \rightarrow +\infty} \left. \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right|_1^b = 2 \lim_{b \rightarrow +\infty} \sqrt{x} \Big|_1^b \\ &= 2 \lim_{b \rightarrow +\infty} (\sqrt{b} - 1) = +\infty, \text{ integral divergira} \end{aligned}$$

$$\text{c) } \int_0^{+\infty} e^{-ax} dx \quad (a > 0) = \lim_{t \rightarrow +\infty} \int_0^t e^{-ax} dx = \left. \begin{array}{l} -ax = s \\ -a dx = ds \\ dx = -\frac{1}{a} ds \end{array} \right|_{\substack{x=0 \Rightarrow s=0 \\ x=t \Rightarrow s=-at}} =$$

$$= \lim_{t \rightarrow +\infty} \int_0^{-at} e^s \left(-\frac{1}{a} \right) ds = -\frac{1}{a} \lim_{t \rightarrow +\infty} \left. e^s \right|_0^{-at} = -\frac{1}{a} \lim_{t \rightarrow +\infty} (e^{-at} - 1) = \frac{1}{a}$$

$$\text{d) } \int_2^{+\infty} \frac{\ln x}{x} dx \quad \text{Rj. divergira} \quad \text{e) } \int_1^{+\infty} \frac{dx}{x^2(x+1)} \quad \text{Rj. 1-1/2} \quad \text{f) } \int_0^{+\infty} x e^{-x^2} dx \quad \text{Rj. } \frac{1}{2}$$

Zadaci za vježbu

Integrali s beskonačnim granicama

U zadacima 2366 — 2385 izračunati donje nesvojstvene integrale (ili ustanoviti njihovu divergenciju).

$$2366. \int_1^{\infty} \frac{dx}{x^4}. \quad 2367. \int_1^{\infty} \frac{dx}{\sqrt{x}}. \quad 2368. \int_0^{\infty} e^{-ax} dx \quad (a > 0).$$

$$2369. \int_{-\infty}^{\infty} \frac{2x dx}{x^2 + 1}. \quad 2370. \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}. \quad 2371. \int_2^{\infty} \frac{\ln x}{x} dx.$$

$$2372. \int_1^{\infty} \frac{dx}{x^2(x+1)}. \quad 2373. \int_0^{\infty} \frac{x}{(1+x)^3} dx. \quad 2374. \int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2-1}}.$$

$$2375. \int_{a^2}^{\infty} \frac{dx}{x\sqrt{1+x^2}}. \quad 2376. \int_0^{\infty} xe^{-x^2} dx. \quad 2377. \int_0^{\infty} x^3 e^{-x^2} dx.$$

$$2378. \int_0^{\infty} x \sin x dx. \quad 2379. \int_0^{\infty} e^{-\sqrt{x}} dx. \quad 2380. \int_0^{\infty} e^{-x} \sin x dx.$$

$$2381. \int_0^{\infty} e^{-ax} \cos bx dx. \quad 2382. \int_1^{\infty} \frac{\operatorname{arctg} x}{x^2} dx. \quad 2383. \int_0^{\infty} \frac{dx}{1+x^3}.$$

$$2384. \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}. \quad 2385. \int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx.$$

U zadacima 2386 — 2393 ispitati da li su dati integrali konvergentni.

$$2386. \int_0^{\infty} \frac{x}{x^3+1} dx. \quad 2387. \int_1^{\infty} \frac{x^3+1}{x^4} dx.$$

$$2388. \int_0^{\infty} \frac{x^{13}}{(x^5+x^3+1)^3} dx. \quad 2389. \int_1^{\infty} \frac{\ln(x^2+1)}{x} dx.$$

$$2390. \int_0^{\infty} \sqrt{x} e^{-x} dx. \quad 2391. \int_0^{\infty} \frac{x \operatorname{arctg} x}{\sqrt{1+x^4}} dx.$$

$$2392. \int_{e^2}^{\infty} \frac{dx}{x \ln \ln x}. \quad 2393. \int_e^{\infty} \frac{dx}{x (\ln x)^{\frac{3}{2}}}.$$

Rješenja

$$2366. \frac{1}{3}.$$

$$2367. \text{Divergira.} \quad 2368. \frac{1}{a}.$$

$$2369. \text{Divergira.} \quad 2370. \pi.$$

$$2371. \text{Divergira.} \quad 2372. 1 - \ln 2.$$

$$2373. \frac{1}{2}. \quad 2374. \frac{\pi}{4}.$$

$$2375. \ln \frac{\sqrt{a^2+1}+1}{a^2}. \quad 2376. \frac{1}{2}.$$

$$2377. \frac{1}{2}. \quad 2378. \text{Divergira.}$$

$$2379. 2. \quad 2380. \frac{1}{2}.$$

$$2381. \frac{a}{a^2+b^2}, \text{ ako je } a > 0,$$

a ne postoji ako je $a < 0$.

$$2382. \frac{\pi}{4} + \frac{1}{2} \ln 2. \quad 2383. \frac{2\pi}{3\sqrt{3}}.$$

$$2384. \frac{\pi}{2}. \quad 2385. \frac{1}{2} + \frac{\pi}{4}.$$

$$2386. \text{Konvergira.} \quad 2387. \text{Divergira.}$$

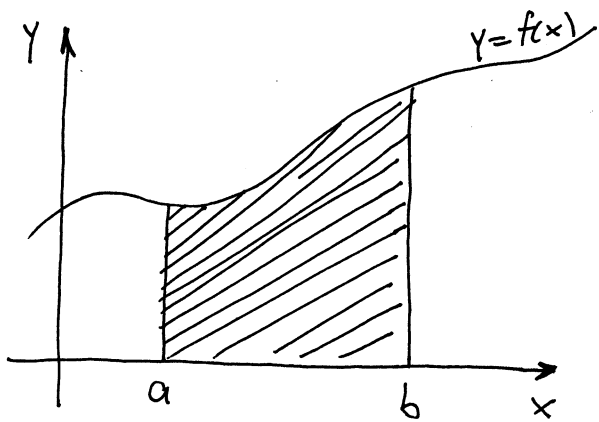
$$2388. \text{Konvergira.} \quad 2389. \text{Divergira.}$$

$$2390. \text{Konvergira.} \quad 2391. \text{Divergira.}$$

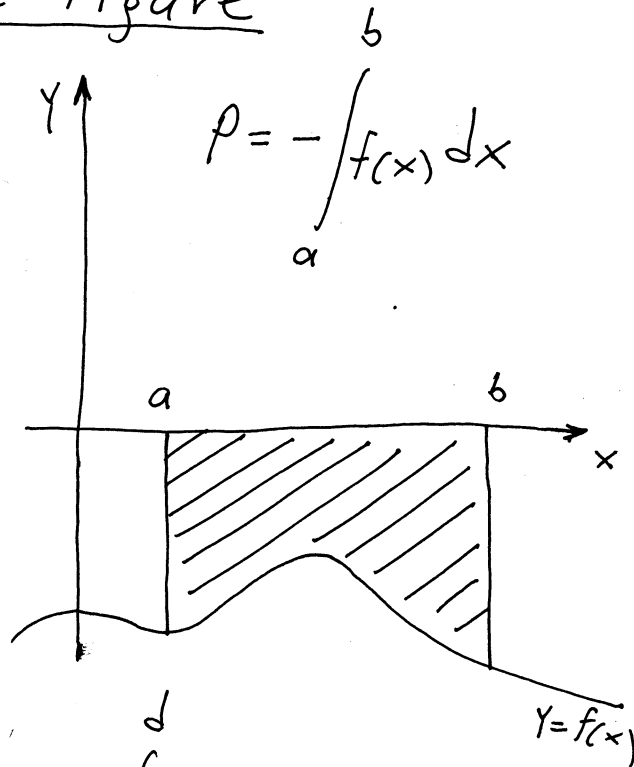
$$2392. \text{Divergira.} \quad 2393. \text{Konvergira.}$$

Primjena određenog integrala

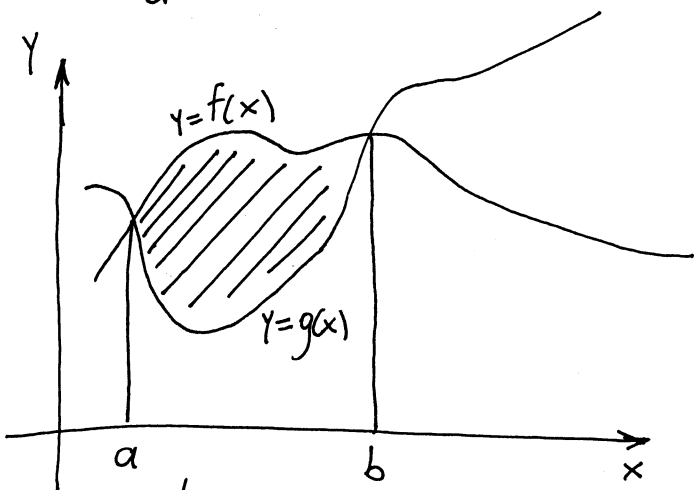
Izračunavanje površine ravne figure



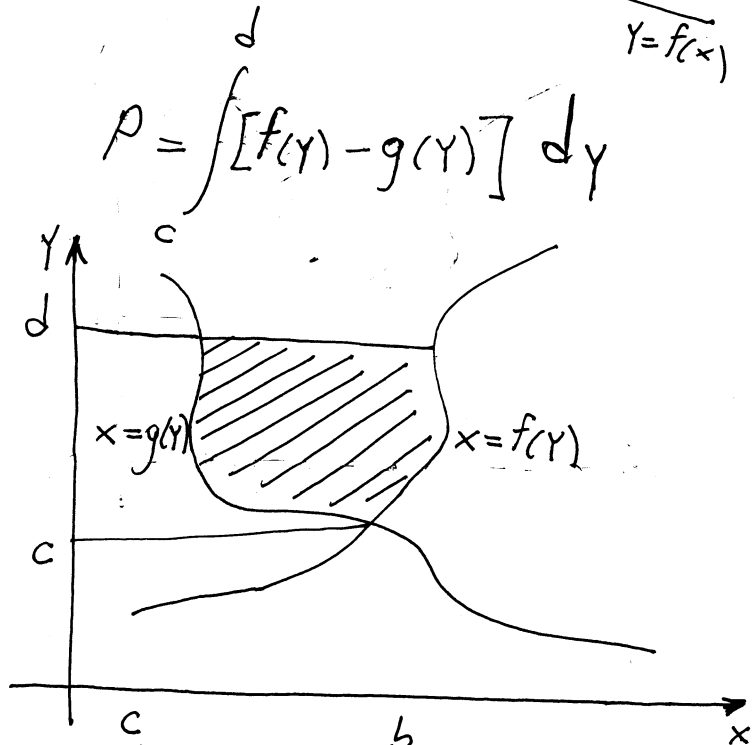
$$P = \int_a^b f(x) dx$$



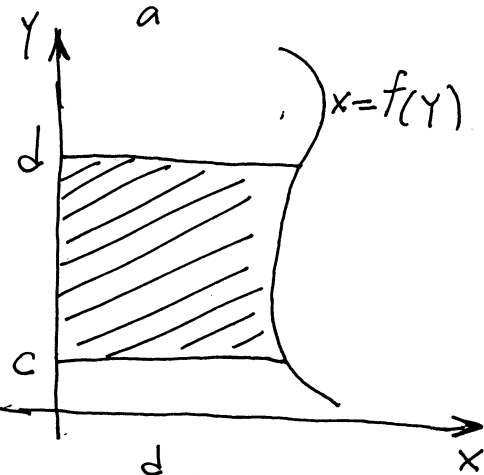
$$P = - \int_a^b f(x) dx$$



$$P = \int_a^b [f(x) - g(x)] dx$$

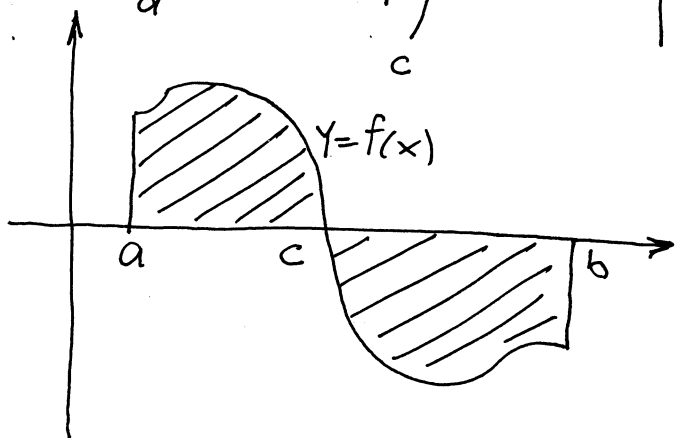


$$P = \int_c^d [f(y) - g(y)] dy$$



$$P = \int_c^d f(y) dy$$

$$P = \int_a^c f(x) dx + \left| \int_c^b f(x) dx \right|$$



1. Izračunati površinu ravne figure koja je ograničena linijama $y=4-(x-2)^2$ i $y=0$.

Rj.

$$y=4-(x-2)^2$$

$$y=4-(x^2-4x+4)$$

$$y=-x^2+4x$$

$$y=-x(x-4)$$

Nule $A(0,0)$ i $B(4,0)$

$$-\frac{b}{2a} = -\frac{4}{2 \cdot (-1)} = 2$$

$$\frac{4ac-b^2}{4a} = \frac{0-16}{-4} = 4$$

Tjeme parabole $y=4-(x-2)^2$ je u tački $(2,4)$.

$$P = \int_0^4 (-x^2+4x) dx = \int_0^4 (-x^2) dx + \int_0^4 4x dx = -\frac{x^3}{3} \Big|_0^4 + 4 \cdot \frac{x^2}{2} \Big|_0^4 = -\frac{1}{3}(4^3-0^3) + 2(4^2-0^2) = -\frac{1}{3} \cdot 64 + 32 = \frac{96}{3} - \frac{64}{3} = \frac{32}{3}$$

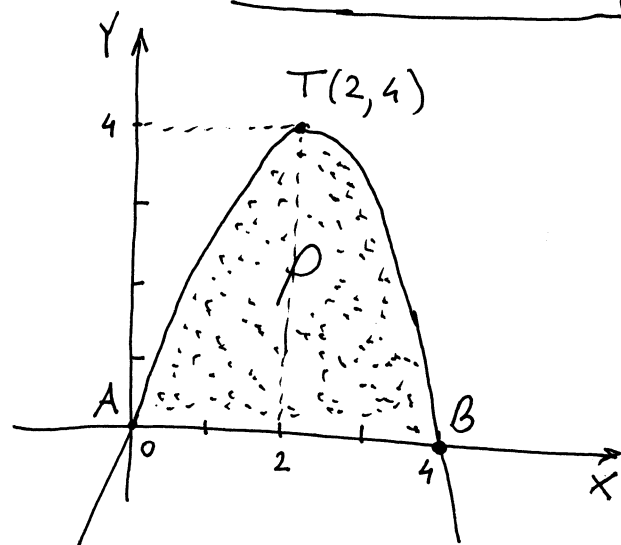
Kriva $y=ax^2+bx+c$ ima grafik u obliku parabole.

Tjeme parabole $T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

za $a > 0$



za $a < 0$



2. Izračunati površinu ravne figure koja je ograničena krivom $y=x^2-4x+3$ i pravama $y=0$, $x=0$ i $x=2$.

Rj. $y=x^2-4x+3$

$$D=16-12=4$$

$$x_{1,2} = \frac{4 \pm 2}{2}$$

Nule krive

$A(1,0)$ i $B(3,0)$

$$-\frac{b}{2a} = -\frac{-4}{2} = 2$$

$$\frac{4ac-b^2}{4a} = \frac{12-16}{4} = -1$$

Tjeme krive $y=x^2-4x+3$ je u tački $T(2,-1)$.

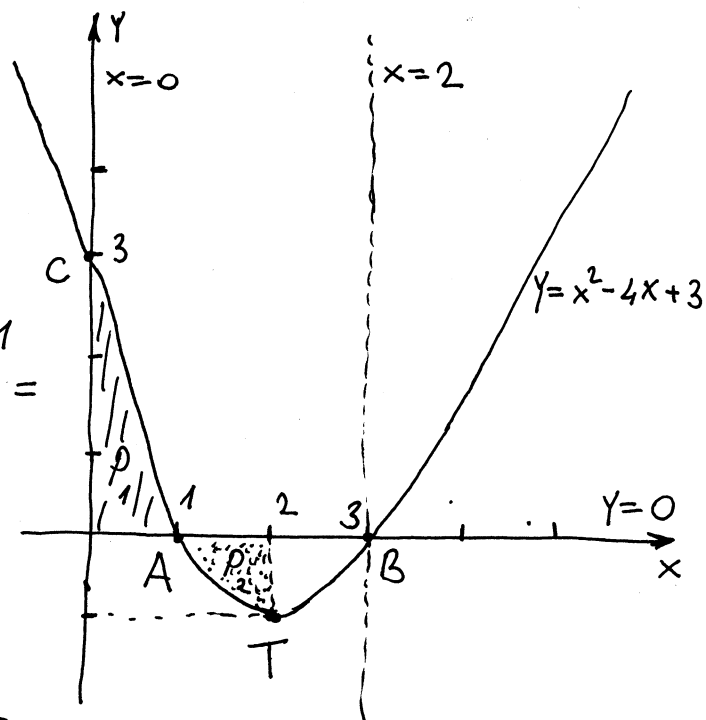
$C(0,3)$ je tačka presjeka
krive sa Y -osom

$$P = P_1 + P_2$$

$$P_1 = \int_0^1 (x^2 - 4x + 3) dx = \left. \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 3x \right|_0^1 =$$

$$= \frac{1}{3}(1-0) - 2(1-0) + 3(1-0) =$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$



$$P_2 = - \int_1^3 (x^2 - 4x + 3) dx = - \left(\left. \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 3x \right|_1^3 \right) = - \left(\frac{1}{3}(8-1) - 2(4-1) + 3 \cdot 1 \right)$$

$$= - \left(\frac{7}{3} - 6 + 3 \right) = - \left(-\frac{2}{3} \right) = \frac{2}{3}$$

$$P = \frac{4}{3} + \frac{2}{3} = 2 \quad \text{tražena površina ravne figure.}$$

③ Izračunati površinu ravne figure kojeg čine
parabola $y = x^2 - 2x + 2$ i prava $x + 2y - 9 = 0$.

Rj.

$$\text{prava } x + 2y - 9 = 0$$

$$2y = -x + 9$$

$$y = -\frac{1}{2}x + \frac{9}{2}$$

prava prolazi kroz

tačke $A(0, \frac{9}{2})$

i $B(9, 0)$.

$$y = x^2 - 2x + 2$$

$$D = 4 - 8 = -4 < 0$$

kriva ne siječe $x=0$
- nema nula

$$x=0 \Rightarrow y=2$$

$C(0, 2)$ je presjek krive sa
 Y -osom

$$-\frac{b}{2a} = -\frac{-2}{2} = 1$$

$T(1, 1)$ je tjeme parabole

$$\frac{4ac - b^2}{4a} = \frac{8 - 4}{4} = 1$$

Trebamo naći još tačke presjeka prave i parabole.

$$y = x^2 - 2x + 2$$

$$x + 2y - 9 = 0$$

$$y = x^2 - 2x + 2$$

$$x = -2y + 9$$

$$y = (-2y + 9)^2 - 2(-2y + 9) + 2$$

$$y_1 = \frac{13}{4} \Rightarrow x = -2 \cdot \frac{13}{4} + 9 = -\frac{13}{2} + \frac{18}{2} = \frac{5}{2}$$

$$y_2 = 5 \Rightarrow x = -2 \cdot 5 + 9 = -1$$

Tačke presjeka prave i parabole

su $R(\frac{5}{2}, \frac{13}{4})$; $Q(-1, 5)$

$$P = \int_{-1}^{\frac{5}{2}} \left[\left(-\frac{1}{2}x + \frac{9}{2}\right) - (x^2 - 2x + 2) \right] dx$$

$$\int_{-1}^{\frac{5}{2}} \left(-\frac{1}{2}x + \frac{9}{2}\right) dx = -\frac{1}{2} \cdot \frac{x^2}{2} \Big|_{-1}^{\frac{5}{2}} + \frac{9}{2}x \Big|_{-1}^{\frac{5}{2}} = -\frac{1}{4} \left(\frac{25}{4} - 1\right) + \frac{9}{2} \left(\frac{5}{2} - (-1)\right)$$

$$= -\frac{1}{4} \cdot \frac{21}{4} + \frac{9}{2} \cdot \frac{7}{2} = \frac{231}{16}$$

$$\int_{-1}^{\frac{5}{2}} (x^2 - 2x + 2) dx = \frac{x^3}{3} \Big|_{-1}^{\frac{5}{2}} - 2 \frac{x^2}{2} \Big|_{-1}^{\frac{5}{2}} + 2x \Big|_{-1}^{\frac{5}{2}} = \frac{1}{3} \left(\frac{125}{8} - (-1)\right) - \left(\frac{25}{4} - 1\right) +$$

$$+ 2 \left(\frac{5}{2} - (-1)\right) = \frac{1}{3} \cdot \frac{133}{8} - \frac{21}{4} + 2 \cdot \frac{7}{2} = \frac{133}{24} + \frac{7}{4} = \frac{175}{24}$$

$$P = \frac{231}{16} - \frac{175}{24} = \frac{231}{4 \cdot 4} - \frac{175}{6 \cdot 4} = \frac{686}{96} = \frac{343}{48}$$

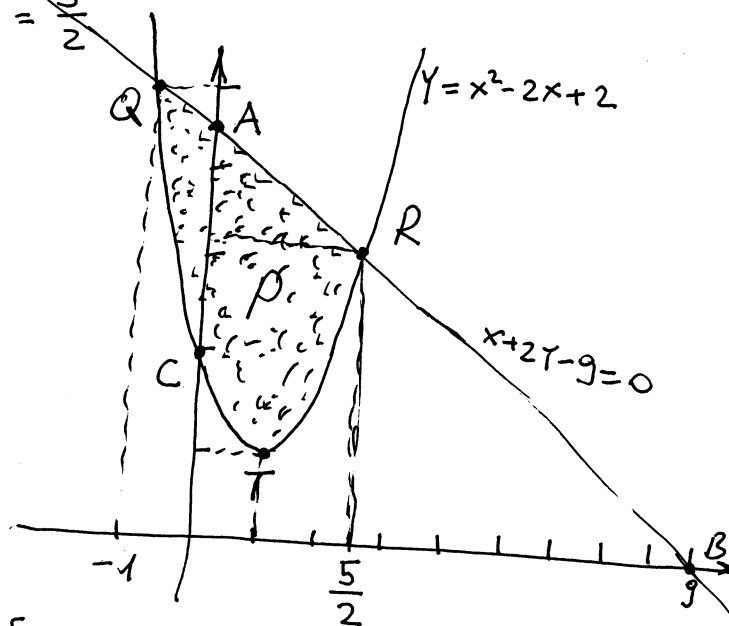
$$y = 4y^2 - 36y + 81 + 4y - 18 + 2$$

$$4y^2 - 32y + 65 = 0$$

$$D = 32^2 - 16 \cdot 65 = 49$$

$$y_{1,2} = \frac{32 \pm 7}{8} \quad y_1 = \frac{26}{8} = \frac{13}{4}$$

$$y_2 = 5$$



4) Izračunati površinu ravne figure koja je ograničena krivom $y^2 = 2x + 1$ i pravom $y = 2x - 1$.
Rj. prava $y = 2x - 1$ prolazi kroz tačke $A(0, -1)$; $B(\frac{1}{2}, 0)$.

$$y^2 = 2x + 1$$

$$2x = y^2 - 1$$

$$x = \frac{1}{2}y^2 - \frac{1}{2}$$

$$x=0 \Rightarrow y^2=1$$

$$A(0, -1); B(0, 1)$$

su tačke presjeka
f-je sa y-osom

$C(-\frac{1}{2}, 0)$ je nula f-je

Kriva oblika $x = ay^2 + bx + c$

ima grafik u obliku parabole



$$a < 0$$



$$a > 0$$

Tjeme krive se traži

po formuli $T(-\frac{D}{4a}, -\frac{b}{2a})$

$$D=1 \quad -\frac{D}{4a} = -\frac{1}{4 \cdot \frac{1}{2}}$$

$$-\frac{b}{2a} = -\frac{0}{2 \cdot \frac{1}{2}} = 0$$

$T(-\frac{1}{2}, 0)$
je tjeme
parabole

Tražimo još tačke presjeka
krive i prave

$$y = 2x - 1$$

$$y^2 = 2x + 1$$

za $x=0$

$$y = -1$$

$$(2x-1)^2 = 2x+1$$

$$4x^2 - 4x + 1 - 2x - 1 = 0$$

$$4x^2 - 6x = 0$$

$$2x(2x-3) = 0$$

$$x=0 \quad \vee \quad x=\frac{3}{2}$$

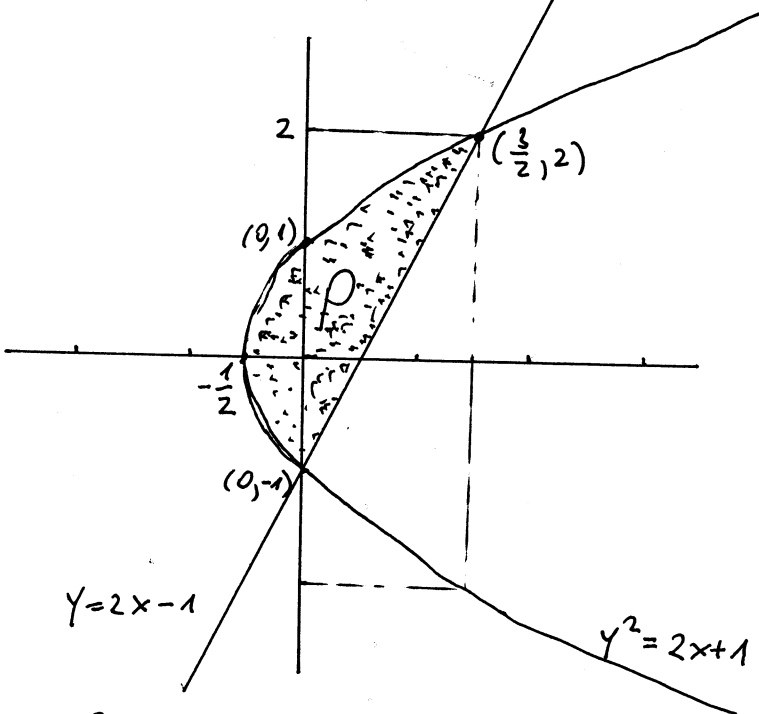
$D(0, -1)$ i $E(\frac{3}{2}, 2)$ su
tačke presjeka krive i prave

$$y = 2x - 1 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}$$

$$y^2 = 2x + 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$$

$$P = \int_{-1}^2 \left[\frac{1}{2}y + \frac{1}{2} - \left(\frac{1}{2}y^2 - \frac{1}{2} \right) \right] dy = \frac{1}{2} \int_{-1}^2 (y + 1 - y^2 + 1) dy = \frac{1}{2} \int_{-1}^2 (-y^2 + y + 2) dy =$$
$$= \frac{1}{2} \cdot \left[\left(-\frac{y^3}{3} \right) \Big|_{-1}^2 + \frac{y^2}{2} \Big|_{-1}^2 + 2y \Big|_{-1}^2 \right] = \frac{1}{2} \left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1) \right] =$$

$$= \frac{1}{2} \left(-3 + \frac{3}{2} + 6 \right) = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4} \quad \text{tražena površina}$$



Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

Rj. $y=1-x^2$

$y(0)=1$

$(0,1)$ je presjek sa y-osom

$1-x^2=0$

$x^2=1$

$x_{1,2}=\pm 1$

$(-1,0)$ i $(1,0)$

su nule f-je

$y=-x^2+1$

parabola
it y loda

$T(-\frac{b}{2a}, -\frac{D}{4a})$

$-\frac{b}{2a} = -\frac{0}{2 \cdot (-1)} = 0$

$D = 0 - 4(-1)(1) = 4$

$-\frac{D}{4a} = -\frac{4}{4 \cdot (-1)} = 1$

$T(0, 1)$

$y-y_1 = y'(x_1)(x-x_1)$

jednačina tangente u tački (x_1, y_1)

$y-y_1 = -\frac{1}{y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1, y_1)

$y' = -2x$ presjek parabole i pozitivnog dijela x-ose je tačka $(1,0)$

$y'(1) = -2$

$y-0 = -\frac{1}{-2}(x-1)$

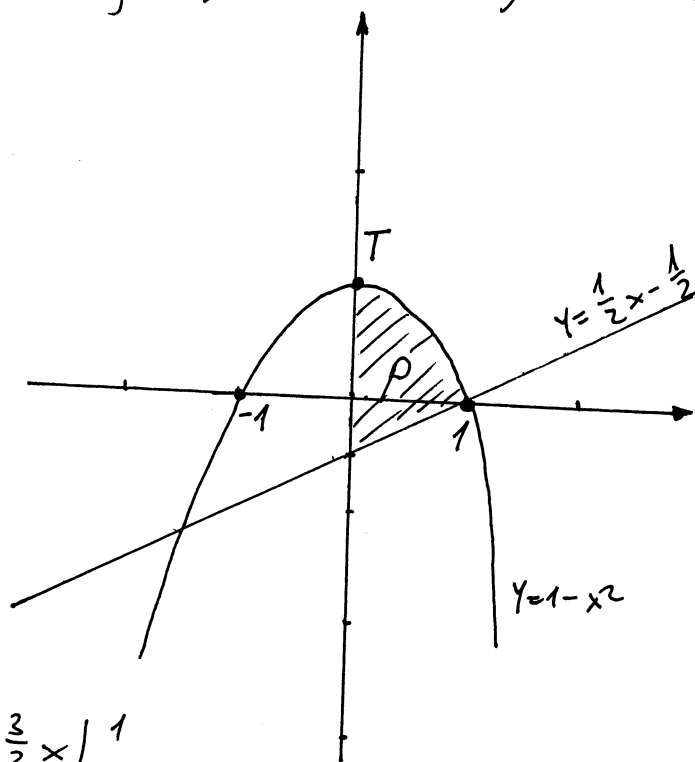
$y = \frac{1}{2}x - \frac{1}{2}$ jednačina normale u tački $(1,0)$

$P = \int_0^1 [(1-x^2) - (\frac{1}{2}x - \frac{1}{2})] dx =$

$= \int_0^1 (-x^2 - \frac{1}{2}x + \frac{3}{2}) dx = -\frac{1}{3}x^3 \Big|_0^1 - \frac{1}{4}x^2 \Big|_0^1 + \frac{3}{2}x \Big|_0^1$

$= -\frac{1 \cdot 4}{3 \cdot 4} - \frac{1 \cdot 3}{4 \cdot 3} + \frac{3}{2} = \frac{3 \cdot 6}{2 \cdot 6} - \frac{7}{12} = \frac{18-7}{12} = \frac{11}{12}$

$P = \frac{11}{12}$ tražena površina



Izračunati površinu koju gradi kriva $y=x^2+x-6$ zajedno sa svojim tangentama povučenim na tu krivu u nul-tačkama krive.

$f: y=x^2+x-6$

$T(-\frac{b}{2a}, -\frac{D}{4a})$ je tjere f-je

$a > 0$

f-je je U oblika

$D=1+24=25$

$-\frac{b}{2a} = -\frac{1}{2}, -\frac{D}{4a} = -\frac{25}{4} = -6\frac{1}{4}$

$Y=(x-2)(x+3)$

$T(-\frac{1}{2}, -6\frac{1}{4})$

$x_1=2, x_2=-3$

$(2,0)$ i $(-3,0)$ su nule f-je

$Y-Y_1=k(x-x_1)$ jednačina prave kroz tačku (x_1, y_1) i koeficijentom k

$f(0)=-6$ tačka

$(0,-6)$ je presjeka f-je sa y-osom

u slučaju tangente $k=Y'(x_1)$

presjek pravih:

$Y'=2x+1$

$(2,0), Y'(2)=5$

$Y=-5x-15$ (1)
 $Y=5x-10$ (2)

$(-3,0), Y'(-3)=-5$

$Y-0=5(x-2)$

(1)+(2): $2Y=-25$

$Y-0=-5(x+3)$

$Y=5x-10$

$Y=-\frac{25}{2}=-12\frac{1}{2}$

jednačina tangente na krivu y u tački $(2,0)$

(1)-(2): $-10x-5=0$

$(-\frac{1}{2}, -12\frac{1}{2})$ je tačka presjeka pravih

$Y=-5x-15$ jednačina tangente na krivu y u tački $(-3,0)$

$-10x=5$

$x=-\frac{1}{2}$

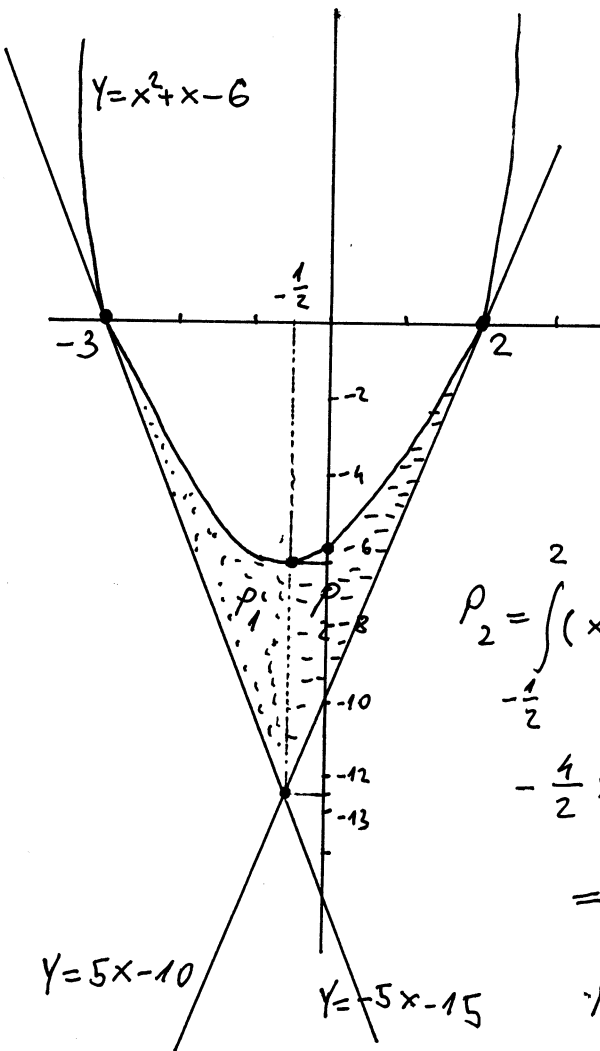
$P=P_1+P_2$

$P_1 = \int_{-3}^{-\frac{1}{2}} (x^2+x-6 - (-5x-15)) dx = \int_{-3}^{-\frac{1}{2}} (x^2+6x+9) dx = \frac{1}{3}x^3 + \frac{6}{2}x^2 + 9x \Big|_{-3}^{-\frac{1}{2}} = \frac{1}{3}(-\frac{1}{8} + 27) + 3(\frac{1}{4} - 9) + 9(-\frac{1}{2} + 3) = \frac{1}{3} \cdot \frac{215}{8} + 3 \cdot \frac{-35}{4} + 9 \cdot \frac{5}{2} = \frac{215}{24} - \frac{630}{24} + \frac{540}{24} = \frac{125}{24}$

$P_2 = \int_{-\frac{1}{2}}^2 (x^2+x-6 - (5x-10)) dx = \int_{-\frac{1}{2}}^2 (x^2-4x+4) dx = \frac{1}{3}x^3 - \frac{4}{2}x^2 + 4x \Big|_{-\frac{1}{2}}^2 = \frac{1}{3}(8 + \frac{1}{8}) - 2(4 - \frac{1}{4}) + 4(2 + \frac{1}{2}) = \frac{1}{3} \cdot \frac{65}{8} - 2 \cdot \frac{15}{4} + 4 \cdot \frac{5}{2} = \frac{65}{24} - \frac{180}{24} + \frac{240}{24} = \frac{125}{24}$

$P = P_1 + P_2 = \frac{125}{24} + \frac{125}{24} = \frac{125}{12}$

tražena površina



Izračunati površinu figure koju ograničavaju linije

$$x = y^2 - 2y - 3 \quad ; \quad y = 3 - 3x$$

Rj. Nađimo presječnu tačku oih linija

$$x = y^2 - 2y - 3$$

$$y = 3 - 3x$$

$$x = (3 - 3x)^2 - 2(3 - 3x) - 3$$

$$x = 9 - 18x + 9x^2 - 6 + 6x - 3$$

$$9x^2 - 13x = 0$$

$$x(9x - 13) = 0$$

$$x = 0 \quad ; \quad ; \quad 9x = 13$$

$$x = \frac{13}{9}$$

$$x = y^2 - 2y - 3$$

$$y = 0 \Rightarrow x = -3$$

$(-3, 0)$ je presjek
krive sa x-osom

$$x = 0 \Rightarrow y = 3$$

$$x = \frac{13}{9} \Rightarrow y = 3 - 3 \cdot \frac{13}{9} = \frac{9}{3} - \frac{13}{3} = -\frac{4}{3}$$

$A(0, 3)$; $B(\frac{13}{9}, -\frac{4}{3})$ su presječne
tačke linija

$x = y^2 - 2y - 3$ je kriva oblika parabole C

čije je tjeme $T(-\frac{D}{4a}, -\frac{b}{2a})$

$$-\frac{b}{2a} = -\frac{-2}{2} = 1, \quad D = 4 + 12 = 16 \quad -\frac{D}{4a} = -\frac{16}{4} = -4$$

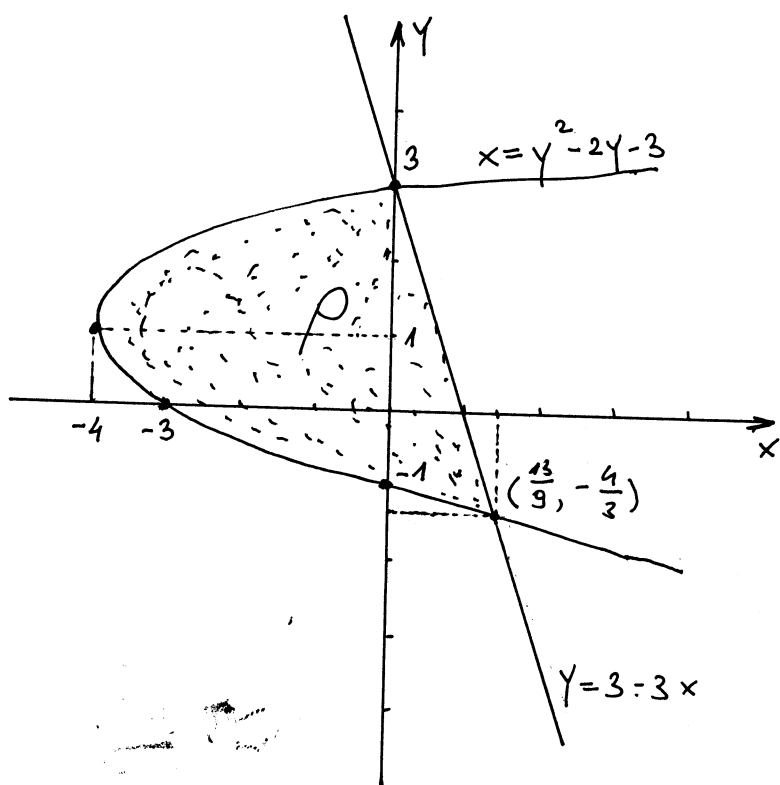
$$T(1, -4)$$

$$y_{1,2} = \frac{2 \pm 4}{2}$$

$$y_1 = \frac{-2}{2} = -1 \quad y_2 = \frac{6}{2} = 3$$

$M_1(0, -1)$; $M_2(0, 3)$

su presjek parabole sa y-osom



$$\rho = \int_{-\frac{4}{3}}^3 \left[\left(1 - \frac{1}{3}y\right) - (y^2 - 2y - 3) \right] dy =$$

$$= \int_{-\frac{4}{3}}^3 (-y^2 + \frac{5}{3}y + 4) dy =$$

$$= -\frac{1}{3}y^3 \Big|_{-\frac{4}{3}}^3 + \frac{5}{3} \cdot \frac{1}{2}y^2 \Big|_{-\frac{4}{3}}^3 + 4y \Big|_{-\frac{4}{3}}^3 =$$

$$= -\frac{1}{3} \left(27 + \frac{64}{27} \right) + \frac{5}{6} \left(9 - \frac{16}{9} \right) + 4 \left(3 + \frac{4}{3} \right)$$

$$= -\frac{1}{3} \cdot \frac{793}{27} + \frac{5}{6} \cdot \frac{65}{9} + 4 \cdot \frac{13}{3} =$$

$$= -\frac{793}{81} + \frac{325}{54} + \frac{52}{3} = \frac{-793 \cdot 2 + 325 \cdot 3 + 52 \cdot 54}{162} = \frac{-1586 + 975 + 2808}{162}$$

$$\rho = \frac{2197}{162} = 13 \frac{91}{162} \quad \text{tražena površina}$$

Izračunati površinu ravne figure koja je ograničena parabolama $y = -x^2 - 4x$ i $y = x^2 + 2x$.

Rj.

Za parabolu $y = -x^2 - 4x$ znamo da je \cap oblika. Vidimo da x-osu siječe u tačkama -4 i 0.

$$y' = -2x - 4$$

$$-2x - 4 = 0$$

$$x = -2$$

Tjeme ove parabole je $T(-2, 4)$

Za parabolu $y = x^2 + 2x$ znamo da je \cup oblika. Vidimo da x-osu siječe u tačkama -2 i 0.

$$y' = 2x + 2$$

$$2x + 2 = 0$$

$$x = -1$$

Tjeme ove parabole je $T(-1, 1)$

Pronađimo još presječne tačke dvije date parabole.

$$y = -x^2 - 4x$$

$$y = x^2 + 2x$$

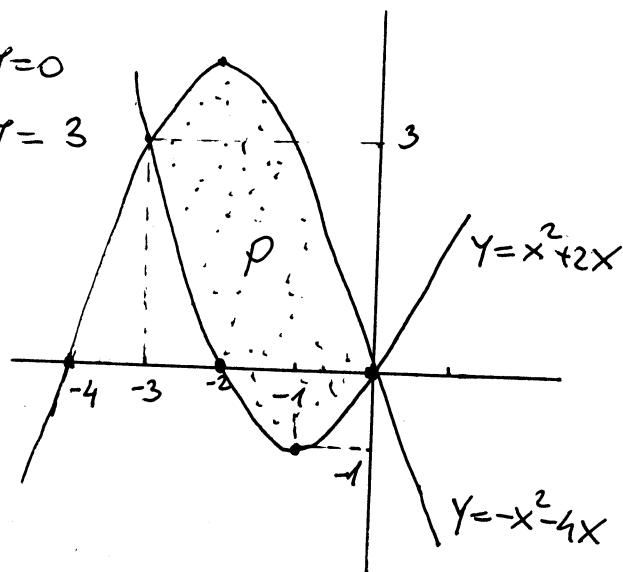
$$-2x^2 - 6x = 0 \quad | :(-2)$$

$$x^2 + 3x = 0$$

$$x(x+3) = 0$$

$$x=0 \Rightarrow y=0$$

$$x=-3 \Rightarrow y=3$$



$$\rho = \int_{-3}^0 [(-x^2 - 4x) - (x^2 + 2x)] dx = \int_{-3}^0 (-2x^2 - 6x) dx = -\frac{2}{3}x^3 \Big|_{-3}^0 - 6 \cdot \frac{1}{2}x^2 \Big|_{-3}^0 =$$

$$= -\frac{2}{3}(0 - (-27)) - 3(0 - 9) = -18 + 27 = 9 \text{ vrijednost tražene površine}$$

Izračunati površinu ravnne figure koja je ograničena krivim linijama $x=y^2-1$ i $x=-y^2-2y+3$

Rj: Za krivu $x=y^2-1$ vidimo da je slijedećeg oblika \subset
 y -osu sječe u tačkama -1 i 1

Kriva $x=-y^2-2y+3$ je oblika \supset . y -osu sječe u tačkama -3 i 1 .

Pronađimo presječne tačke dvije date krive.

$$x = y^2 - 1$$

$$y = -2 \Rightarrow x = 3$$

$$x = -y^2 - 2y + 3$$

$$y = 1 \Rightarrow x = 0$$

$$2y^2 + 2y - 4 = 0 \quad |:2$$

$$y^2 + y - 2 = 0$$

$$(y+2)(y-1) = 0$$

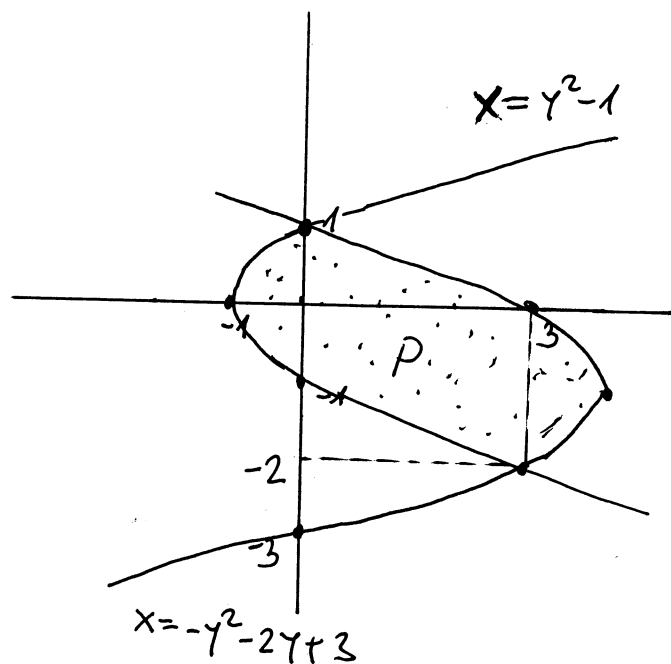
$$P = \int_{-2}^1 [(-y^2 - 2y + 3) - (y^2 - 1)] dy =$$

$$= \int_{-2}^1 (-2y^2 - 2y + 4) dy = -\frac{2}{3}y^3 \Big|_{-2}^1 - y^2 \Big|_{-2}^1 + 4y \Big|_{-2}^1 =$$

$$= -\frac{2}{3}(1 - (-8)) - (1 - 4) + 4(1 - (-2)) =$$

$$= -6 + 3 + 12 = 9 \quad \text{trajnaing}$$

površing



⊕ Izračunati površinu ravne figure koja je ograničena parabolama $x=y^2-4y+3$ i $x=-y^2+2y+3$,

Rj. Parabola $x=y^2-4y+3$ je C oblika, $x=(y-3)(y-1)$
 y -osu siječe u tačkama 1 i 3,

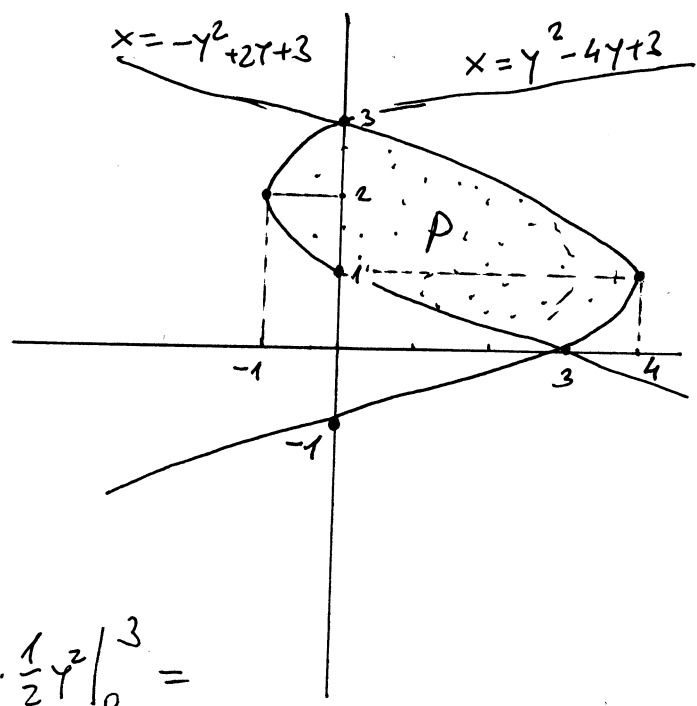
$$\begin{aligned} x' &= 2y-4 \\ 2y-4 &= 0 \\ y &= 2 \end{aligned} \quad x = 4-8+3 = -1 \quad \text{Tjeme ove parabole je } T(-1, 2)$$

Parabola $x=-y^2+2y+3$ je D oblika, $x=-(y+1)(y-3)$
 y -osu siječe u tačkama -1 i 3.

$$\begin{aligned} x' &= -2y+2 \\ -2y+2 &= 0 \\ y &= 1 \end{aligned} \quad x = -1+2+3 = 4 \quad \text{Tjeme ove parabole je } T(4, 1)$$

Pronađimo još presječne tačke dvije date parabole

$$\begin{array}{r} x = y^2 - 4y + 3 \\ x = -y^2 + 2y + 3 \\ \hline 2y^2 - 6y = 0 \\ 2y(y-3) = 0 \end{array} \quad \begin{array}{l} y=0 \Rightarrow x=3 \\ y=3 \Rightarrow x=0 \end{array}$$

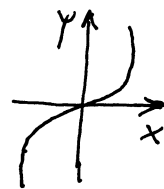


$$\begin{aligned} P &= \int_0^3 [(-y^2+2y+3) - (y^2-4y+3)] dy = \\ &= \int_0^3 (-2y^2+6y) dy = -\frac{2}{3}y^3 \Big|_0^3 + 6 \cdot \frac{1}{2}y^2 \Big|_0^3 = \\ &= -2 \cdot 9 + 3 \cdot 9 = 9 \quad \text{tražena površina} \end{aligned}$$

Izračunati površinu figure koja je određena linijama $y = -x$, $y = \sqrt[3]{x}$, $y = 3x - 2$.

R: Grafički nije teško predstaviti prave $y = -x$ i $y = 3x - 2$. Problem predstavlja kriva $y = \sqrt[3]{x}$.

Ako znamo da kriva $y = x^3$ izgleda ovako



Onda nije teško nacrtati krivu $x = y^3$ što je ekvivalentno sa $y = \sqrt[3]{x}$.

Pronađimo tačke preseka datih krivih.

$$\begin{array}{l} y = -x \\ y = 3x - 2 \end{array}$$

$$-x = 3x - 2$$

$$-4x = -2$$

$$x = \frac{1}{2} \Rightarrow y = -\frac{1}{2}$$

$$\begin{array}{l} y = -x \\ y = \sqrt[3]{x} \end{array}$$

$$y = -x$$

$$y^3 = x$$

$$-x^3 = x$$

$$x^3 + x = 0$$

$$x(x^2 + 1) = 0$$

$$x = 0 \Rightarrow y = 0$$

$$\begin{array}{l} y = 3x - 2 \\ y = \sqrt[3]{x} \end{array}$$

$$\sqrt[3]{x} = 3x - 2$$

$$(3x - 2)^3 = x$$

$$27x^3 - 3 \cdot (3x)^2 \cdot 2 +$$

$$+ 3 \cdot 3x \cdot (-2)^2 + (-2)^3 = x$$

$$27x^3 - 54x^2 + 36x - 8 = x$$

$$27x^3 - 54x^2 + 35x - 8 = 0$$

pokušajmo riješiti sistem na drugi način

$$\sqrt[3]{x} = 3x - 2$$

$$x = t^3$$

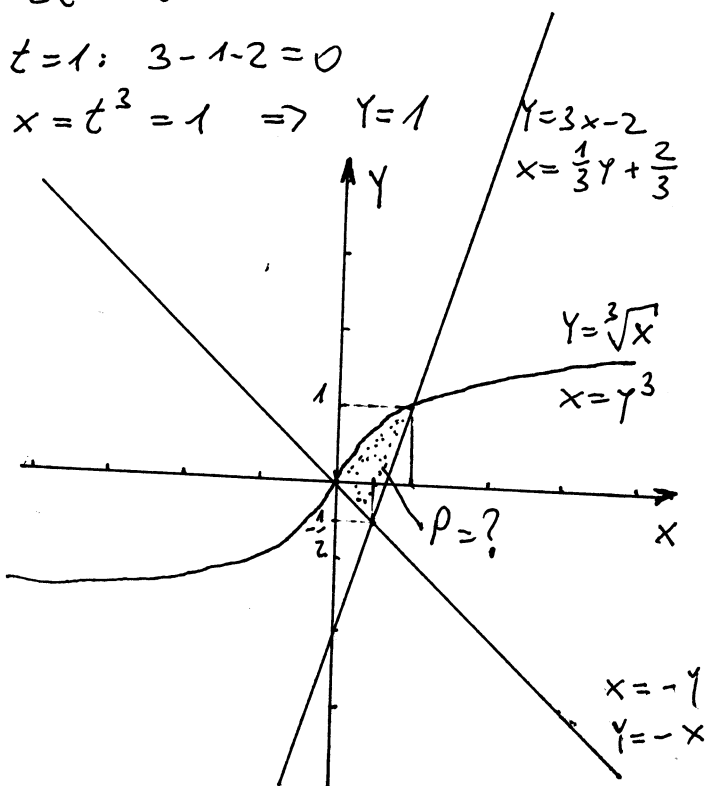
$$3t^3 - 2 = t$$

$$3t^3 - t - 2 = 0$$

$$t = 1: 3 - 1 - 2 = 0$$

$$x = t^3 = 1 \Rightarrow y = 1$$

$$\left[\begin{array}{l} 3x = y + 2 \\ \end{array} \right]$$



$$P = \int_{-\frac{1}{2}}^1 \left[\left(\frac{1}{3}y + \frac{2}{3} \right) - (-y) \right] dy + \int_{-\frac{1}{2}}^0 \left[\left(\frac{1}{3}y + \frac{2}{3} \right) - y^3 \right] dy =$$

$$= \int_{-\frac{1}{2}}^1 \left(\frac{4}{3}y + \frac{2}{3} \right) dy + \int_{-\frac{1}{2}}^0 \left(-y^3 + \frac{1}{3}y + \frac{2}{3} \right) dy =$$

$$= \frac{4}{3} \cdot \frac{1}{2} y^2 \Big|_{-\frac{1}{2}}^1 + \frac{2}{3} y \Big|_{-\frac{1}{2}}^1 - \frac{1}{4} y^4 \Big|_0^1 + \frac{1}{3} \cdot \frac{1}{2} y^2 \Big|_0^1$$

$$+ \frac{2}{3} y \Big|_0^1 = \frac{2}{3} \cdot \left(-\frac{1}{4} \right) + \frac{2}{3} \cdot \frac{1}{2} - \frac{1}{4} + \frac{1}{6} +$$

$$+ \frac{2}{3} = -\frac{1}{6} + \frac{1}{3} - \frac{1}{4} + \frac{1}{6} + \frac{2}{3} = \frac{3}{4}$$

Izračunati površinu figure koja je određena linijama $Y=-2$, $Y=x^3+x$, $x+Y=3$.

Rj. $Y=-2$, $x+Y=3$ su prave linije i njih nije teško nacrtati. Problem za crtanje predstavlja kriva $Y=x^3+x$.

Ispitajmo f-ju $Y=x^3+x$. D: $x \in \mathbb{R}$

$f(-x) = -x^3 - x = -(x^3 + x)$ f-ja je neparna

$A(0,0)$ je nula f-je i presjek sa y-osom

f-ja nema prekida \Rightarrow f-ja nema vertikalnu asimptotu

f-ja nema horizontalnu ni kosu asimptotu

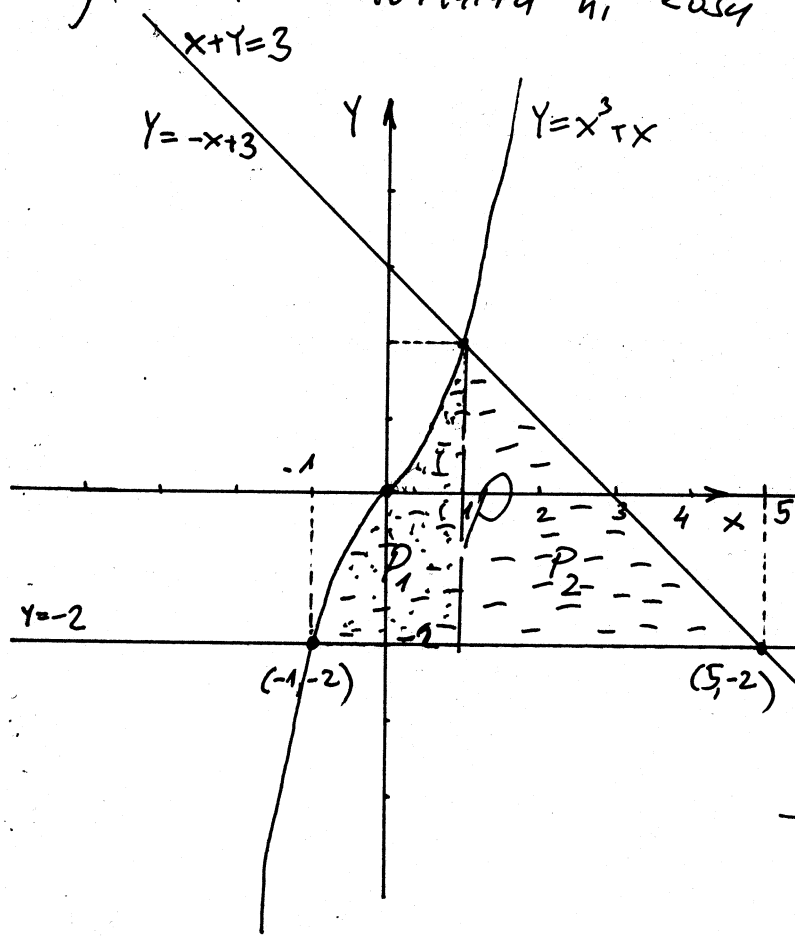
$Y' = 3x^2 + 1$ f-ja je uvijek pozitivna (vaste za svako x)

f-ja nema ekstrem

$Y'' = 6x$

x	(0, + ∞)
Y''	+
Y	U

(0,0) je manji težište



f-ja je ovog oblika

Nadimo tačke presjeka datih krivih.

$$\begin{array}{r} y = -2 \\ x + y = 3 \\ \hline x - 2 = 3 \\ x = 5 \end{array}$$

$(5, -2)$ je tačka presjeka

$$\begin{array}{r} y = -2 \\ y = x^3 + x \\ \hline -2 = x^3 + x \\ x^3 + x + 2 = 0 \\ x = -1: -1 - 1 + 2 = 0 \end{array}$$

$$x^3 + x + 2 = (x+1)(x^2 - x + 2)$$

$> 0 \forall x$

Rješenje jednačine $x^3 + x + 2 = 0$ je $x = -1$.

$(-1, -2)$ je tačka presjeka datih krivih

$$\begin{array}{r} (x^3 + x + 2) : (x+1) = x^2 - x + 2 \\ - \underline{x^3 + x^2} \\ -x^2 + x + 2 \\ - \underline{-x^2 - x} \\ 2x + 2 \\ \underline{2x + 2} \\ // \end{array}$$

$$Y = x^3 + x$$

$$x + y = 3$$

$$Y = x^3 + x$$

$$Y = -x + 3$$

$$-x + 3 = x^3 + x$$

$$x^3 + 2x - 3 = 0$$

$$x=1: 1^3 + 2 \cdot 1 - 3 = 3 - 3 = 0$$

$$(x^3 + 2x - 3) : (x - 1) = x^2 + x + 3$$

$$\begin{array}{r} x^3 - x^2 \\ \hline x^2 + 2x - 3 \\ - x^2 - x \\ \hline 3x - 3 \\ - 3x - 3 \\ \hline = = \end{array}$$

$$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{> 0 \forall x} (x - 1)$$

(1, 2) je presječna
tačka krivih

$$P_1 = \int_{-1}^1 [(x^3 + x) - (-2)] dx = \int_{-1}^1 (x^3 + x + 2) dx = \left. \frac{1}{4} x^4 \right|_{-1}^1 + \left. \frac{1}{2} x^2 \right|_{-1}^1 + \left. 2x \right|_{-1}^1 = 4$$

$$P_2 = \int_1^5 [(-x + 3) - (-2)] dx = \int_1^5 (-x + 5) dx = -\left. \frac{x^2}{2} \right|_1^5 + \left. 5x \right|_1^5 = -\frac{1}{2}(25 - 1) + 5 \cdot 4 = -\frac{1}{2} \cdot 24 + 20 = 20 - 12 = 8$$

$$P = P_1 + P_2 = 8 + 4 = 12 \text{ površina figure}$$

Izračunati površinu figure koju čine linije

$$y = (x-1)^2, \quad \frac{x^2}{1} - \frac{y^2}{2} = 1.$$

Rj. Da bi odredili granice za računanje površine potrebno je grafički predstaviti ove dvije linije.

ispitajmo f-ju $y = (x-1)^2$

D: $x \in \mathbb{R}$

f-ja nije ni parna ni neparna

$f(0) = 1$, $(0, 1)$ je presjek sa y-osi

$(x-1)^2 = 0 \Rightarrow x = 1$, $(1, 0)$ je nula f-je

$y = (x-1)^2 = x^2 - 2x + 1 \Rightarrow$ f-ja je oblika

Nadamo još breme f-je

$$y' = 2x - 2$$

$$y' = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$$

$T(1, 0)$

Kako je $g(1) = 0 \Rightarrow g(x)$ je djeljivo sa $(x-1)$

$$(x^4 - 4x^3 + 4x^2 - 4x + 3) : (x-1) = x^3 - 3x^2 + x - 3$$

$$\begin{array}{r} x^4 - 4x^3 + 4x^2 - 4x + 3 \\ - (x^4 - x^3) \\ \hline -3x^3 + 4x^2 - 4x + 3 \\ - (-3x^3 + 3x^2) \\ \hline x^2 - 4x + 3 \\ - (x^2 - x) \\ \hline -3x + 3 \\ - (-3x + 3) \\ \hline = = \end{array}$$

$$g(x) = (x^3 - 3x^2 + x - 3)(x-1) = g_1(x)$$

$$g_1(0) = -3$$

$$g_1(1) = 1 - 3 + 1 - 3 = -4$$

$$g_1(2) = 8 - 12 + 2 - 3 = -5$$

$$g_1(3) = 27 - 27 + 3 - 3 = 0$$

$$g_1(-2) = -8 - 12 - 2 - 3 = -25$$

$$g_1(-1) = -1 - 3 - 1 - 3 = -8$$

$$g_1(-3) = -27 - 27 - 3 - 3 = -60$$

$\Rightarrow g_1(x)$ je djeljivo sa $x-3$

$$(x^3 - 3x^2 + x - 3) : (x-3) = x^2 + 1$$

$$\begin{array}{r} x^3 - 3x^2 + x - 3 \\ - (x^3 - 3x^2) \\ \hline x - 3 \\ - (x - 3) \\ \hline = = \end{array}$$

Prema tome $g(x) = (x^2 + 1)(x-3)(x-1)$

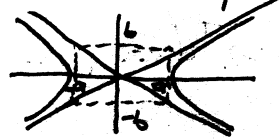
Za $x = 3 \Rightarrow y = 4$

Za $x = 1 \Rightarrow y = 0$

Presjecne tačke krivih su $(3, 4)$ i $(1, 0)$

Krivce oblika

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ zovemo HIPERBOLE i one su oblika}$$



Prije nego grafički predstavimo liniju $y = (x-1)^2$ pronađimo u kojim tačkama siječe liniju $\frac{x^2}{1} - \frac{y^2}{2} = 1$.

$$y = x^2 - 2x + 1 = (x-1)^2$$

$$2x^2 - y^2 = 2$$

$$y^2 = (x^2 - 2x + 1)^2 =$$

$$= (x^2 - 2x + 1)(x^2 - 2x + 1) =$$

$$= x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x + 1$$

$$= x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$2x^2 - y^2 = 2$$

$$\frac{2x^2 - x^4 + 4x^3 - 6x^2 + 4x - 1 - 2 = 0}{(6-1)}$$

$$x^4 - 4x^3 + 4x^2 - 4x + 3 = 0$$

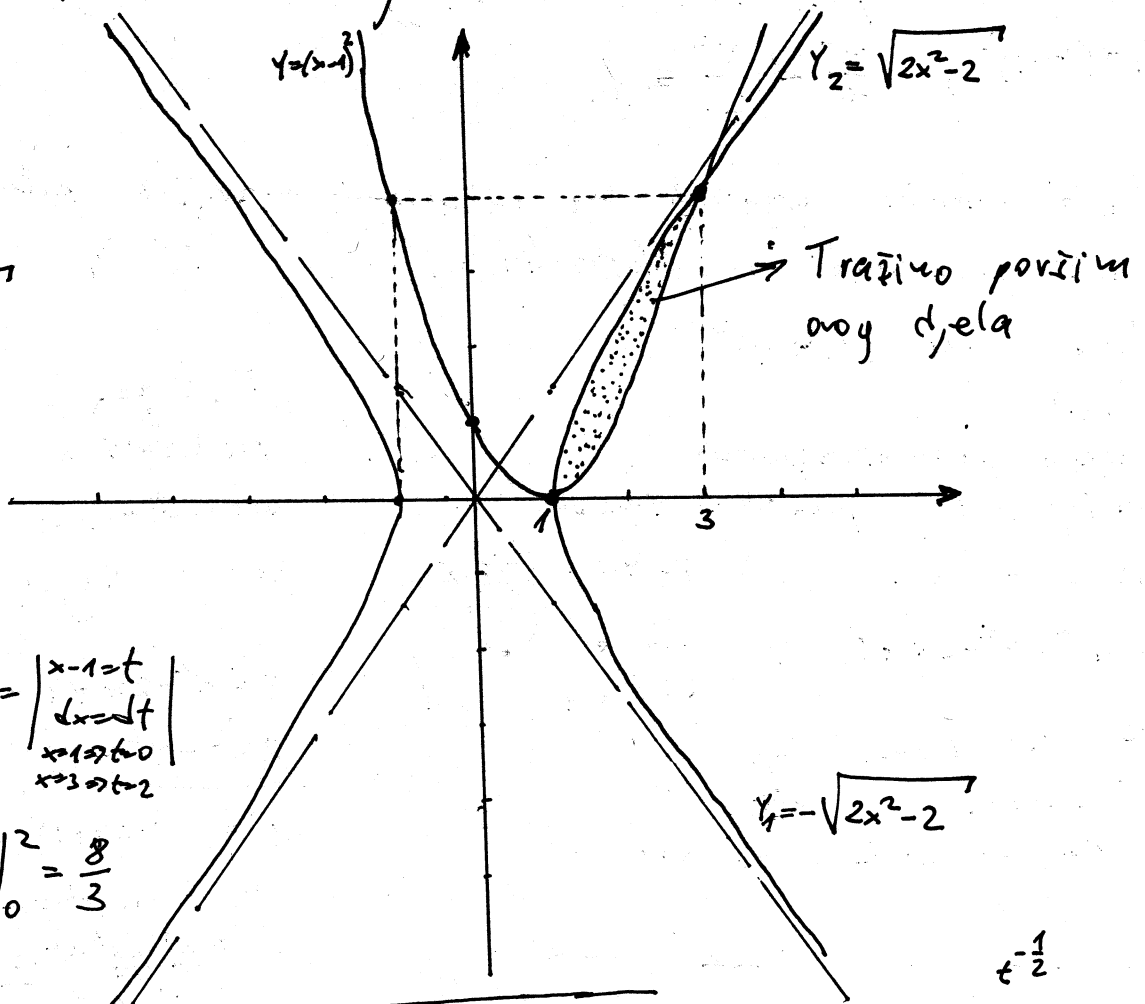
Označimo ovo sa $g(x) = x^4 - 4x^3 + 4x^2 - 4x + 3$

Nacrtejmo naše krive linije

$$\sqrt{2}x, 41$$

$$y^2 = 2x^2 - 2$$

$$y_{1,2} = \pm \sqrt{2x^2 - 2}$$



$$P_2 = \int_1^3 (x-1)^2 dx = \left| \begin{array}{l} x-1=t \\ dx=dt \\ x=1 \rightarrow t=0 \\ x=3 \rightarrow t=2 \end{array} \right|$$

$$= \int_0^2 t^2 dt = \frac{1}{3} t^3 \Big|_0^2 = \frac{8}{3}$$

$$P = \int_1^3 (\underbrace{\sqrt{2x^2-2}}_{P_1} - \underbrace{(x-1)^2}_{P_2}) dx$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \left| \begin{array}{l} x^2-1=t \\ 2x dx = dt \\ x dx = \frac{1}{2} dt \end{array} \right| = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \sqrt{t} + C = \sqrt{x^2-1} + C$$

$$P_1 = \int_1^3 \sqrt{2} \cdot \sqrt{x^2-1} dx = \sqrt{2} \int_1^3 \frac{x^2-1}{\sqrt{x^2-1}} dx = \sqrt{2} \left(\int_1^3 \frac{x^2}{\sqrt{x^2-1}} dx - \int_1^3 \frac{dx}{\sqrt{x^2-1}} \right)$$

$$\int_1^3 x \cdot \frac{x}{\sqrt{x^2-1}} dx = \left| \begin{array}{l} u=x \\ du=dx \\ dv = \frac{x}{\sqrt{x^2-1}} dx \\ v = \sqrt{x^2-1} \end{array} \right| = \frac{x \sqrt{x^2-1}}{3\sqrt{8}-0} \Big|_1^3 - \int_1^3 \sqrt{x^2-1} dx$$

$$\int_1^3 \frac{dx}{\sqrt{x^2-1}} = \ln|x + \sqrt{x^2-1}| \Big|_1^3 = \ln|3 + \sqrt{8}|$$

$$\sqrt{2} \int_1^3 \sqrt{x^2-1} dx = \sqrt{2} \cdot \frac{3\sqrt{8}}{6\sqrt{2}} - \sqrt{2} \int_1^3 \sqrt{x^2-1} dx - \sqrt{2} \ln(3 + 2\sqrt{2})$$

$$\int_1^3 \sqrt{2x^2-2} dx = 6 - \frac{\sqrt{2} \ln(2\sqrt{2}+3)}{2}$$

$$P = P_1 - P_2 = \frac{10}{3} - \frac{\sqrt{2} \ln(2\sqrt{2}+3)}{2}$$

tražena površina

Zadaci za vježbu

Površina ravne figure

2455. Izračunati površinu ograničenu linijama $y^2 = 2x + 1$ i $x - y - 1 = 0$.

2456. Izračunati površinu ograničenu parabolom $y = -x^2 + 4x - 3$ i njenim tangentama u tačkama $(0; 3)$ i $(3; 0)$.

2457. Izračunati površinu ograničenu parabolom $y^2 = 2px$ i njenom normalom koja sa x -osom zaklapa ugao od 135° .

2458. Izračunati površinu ograničenu parabolama $y = x^2$ i $y = \sqrt{x}$.

2459. Izračunati površinu ograničenu parabolama $y^2 + 8x + 16$ i $y^2 - 24x = 48$.

2460. Izračunati površinu ograničenu parabolama $y = x^2$ i $y = \frac{x^3}{3}$.

2461. Parabola $y = \frac{x^2}{2}$ deli krug $x^2 + y^2 = 8$ na dva dela; naći površinu

svakog od njih.

2462. Parabola $y^2 = 6x$ deli krug $x^2 + y^2 = 16$ na dva dela; naći njihove površine.

2463. Iz kruga poluprečnika a isečena je elipsa čija je velika osa jednaka jednom od prečnika kruga, a manja osa je $2b$. Dokazati da je površina preostalog dela kruga jednaka površini elipse čije su poluose a i $a - b$.

2464. Naći površinu ograničenu lukom hiperbole i njenom tetivom koja prolazi kroz žižu i normalna je na realnu osu.

2465. Hiperbola $x^2 - 2y^2 = \frac{a^2}{4}$ deli krug $x^2 + y^2 = a^2$ na tri dela; naći površinu svakog od njih.

2466. Hiperbola $\frac{x^2}{2} - y^2 + 1$ deli oblast ograničenu elipsom $\frac{x^2}{4} + y^2 = 1$ na tri dela; izračunati njihove površine.

2467. Izračunati površinu ograničenu krivama $y = \frac{1}{1+x^2}$ i $y = \frac{x^2}{2}$.

2468. Izračunati površinu ograničenu krivom $y = x(x-1)^2$ i apscisnom osom.

2469. Naći površinu ograničenu krivom $x = y^2(y-1)$ i ordinatnom osom.

2470. Krive zadate jednačinama $y^m = x^n$ i $y^n = x^m$ u kojima su m i n celi pozitivni brojevi, ograničavaju jednu ili više (zatvorenih) oblasti. Izračunati površinu one od njih koja leži u prvom kvadrantu, a zatim i ukupnu površinu ograničenu ovim linijama vodeći računa o parnosti i neparnosti brojeva m i n .

2471. a) Izračunati površinu ograničenu krivom $y = x - x^2\sqrt{x}$ i x -osom. b) Izračunati površinu ograničenu dvema granama krive $(y-x)^2 = x^3$ i pravom $x = 4$.

2472. Izračunati površinu oblasti ograničene krivom $(y-x-2)^2 = 9x$ i koordinatnim osama.

2473. Izračunati površinu ograničenu petljom krive $y^2 = x(x-1)^2$.

2474. Izračunati površinu ograničenu zatvorenom krivom $y^2 = (1-x^2)^3$.

2475. Izračunati površinu ograničenu zatvorenom krivom $y^2 = x^2 - x^4$.

2476. Izračunati površinu ograničenu zatvorenom krivom $x^4 - ax^3 + a^2y^2 = 0$.

2477. Izračunati površinu konačne oblasti ograničene krivom $x^2y^2 = -4(x-1)$ i pravom koja prolazi kroz njene prevojne tačke.

2478. Izračunati površinu ograničenu krivama $y = e^x$, $y = e^{-x}$ i pravom $x = 1$.

2479. Izračunati površinu konačne oblasti ograničene krivom $y = (x^2 + 2x)e^{-x}$ i x -osom.

2480. Izračunati površinu ograničenu krivom $y = e^{-x}(x^2 + 3x + 1) + e^2$, x -osom i dvema pravama, paralelnim y -osi, koje prolaze kroz tačke ekstremuma funkcije y .

Rješenja

2455. $\frac{16}{3}$. 2456. $\frac{9}{4}$. 2457. $\frac{16}{3}p^2$.

2458. $\frac{1}{3}$. 2459. $\frac{32}{3}\sqrt{6}$.

2460. $2\frac{1}{4}$. 2461. $2\pi + \frac{4}{3}$ i $6\pi - \frac{4}{3}$.

2462. $\frac{4}{3}(4\pi + \sqrt{3})$ i $\frac{4}{3}(8\pi - \sqrt{3})$.

2464. $\frac{b^2c}{a} - ab \ln \frac{c+b}{a} - b [eb - a \ln (e + \sqrt{e^2 - 1})]$,

gde je e ekscentricitet.

2465. $a^2 \left[\frac{\pi}{6} - \frac{\sqrt{2}}{8} \ln(\sqrt{3} + \sqrt{2}) \right]$;

$a^2 \left[\frac{\pi}{6} - \frac{\sqrt{2}}{8} \ln(\sqrt{3} + \sqrt{2}) \right]$

i $a^2 \left[\frac{2\pi}{3} + \frac{\sqrt{2}}{4} \ln(\sqrt{3} + \sqrt{2}) \right]$.

2466. $S_1 - S_2 - \pi - \frac{\sqrt{2}}{2} \ln 3 - 2 \arcsin \sqrt{\frac{2}{3}} \approx 0,46$.

$S_2 - 2(\pi - S_1)$, 2467. $\frac{\pi}{2} - \frac{1}{3}$.

2468. $\frac{1}{12}$. 2469. $\frac{1}{12}$.

2470. $\left| \frac{m-n}{m+n} \right|$; $4 \left| \frac{m-n}{m+n} \right|$,

ako su brojevi m

i n oba parni; $2 \left| \frac{m-n}{m+n} \right|$,

ako su m i n — oba neparni;

$\left| \frac{m-n}{m+n} \right|$, ako je jedan od

brojeva m i n paran a drugi neparan.

2471. a) $\frac{3}{14}$; b) $73\frac{1}{7}$.

2472. 1. (oblast se sastoji iz dva dela čije su površine jednake među sobom.

2473. $\frac{8}{15}$. 2474. $\frac{3}{4}\pi$.

2475. $\frac{4}{3}$. 2476. $\frac{\pi a^2}{8}$.

2477. $8 \left(\sqrt{1 + \frac{2}{3}\sqrt{3}} - \arctg \sqrt{1 + \frac{2}{3}\sqrt{3}} \right)$.

2478. $e + \frac{1}{e} - 2$.

2479. 4. 2480. $\frac{3}{e}(e^2 - 4)$.